
Integral Equations

Complete Worked Solutions

Companion to *An Interactive Course in Theory, Methods, and Computation*

$$u(x) = f(x) + \lambda \int_a^b R(x, t; \lambda) f(t) dt$$

GAURAV TIWARI

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Companion to *Integral Equations: An Interactive Course in Theory, Methods, and Computation*
by Gaurav Tiwari

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About These Solutions

Every numbered exercise in the book has a full solution here, in the order and with the numbering used in the book. References such as “Theorem 12.4” or “(10.7)” point to the book. The book itself prints a hint or answer for every exercise and a small set of these solutions in full.

The solutions are written to be read after an honest attempt. When an exercise asks for a proof, the solution gives one complete argument; other correct arguments exist. When an exercise asks for a computation, the solution states the method, the key intermediate results and a check in the original equation.

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Chapter 1 Why Integral Equations?

Exercise 1.1. Integrate by parts with $U = x - t$ and $dV = \cos t \, dt$, so that $dU = -dt$ and $V = \sin t$:

$$\int_0^x (x - t) \cos t \, dt = \left[(x - t) \sin t \right]_0^x + \int_0^x \sin t \, dt = 0 + (1 - \cos x).$$

Hence $1 - \int_0^x (x - t) \cos t \, dt = 1 - (1 - \cos x) = \cos x$, which is the left-hand side. The identity holds for every real x .

Exercise 1.2. Setting $x = 0$ gives $u(0) = 0$. A continuous solution makes the right-hand side differentiable, so $u'(x) = 1 + \lambda u(x)$. For $\lambda \neq 0$, multiply by $e^{-\lambda x}$: $(e^{-\lambda x} u)' = e^{-\lambda x}$, so $e^{-\lambda x} u(x) = \int_0^x e^{-\lambda s} \, ds = (1 - e^{-\lambda x})/\lambda$ and $u(x) = (e^{\lambda x} - 1)/\lambda$. For $\lambda = 0$ the equation reduces to $u(x) = x$.

Check for $\lambda \neq 0$:

$$x + \lambda \int_0^x \frac{e^{\lambda t} - 1}{\lambda} \, dt = x + \int_0^x (e^{\lambda t} - 1) \, dt = x + \frac{e^{\lambda x} - 1}{\lambda} - x = \frac{e^{\lambda x} - 1}{\lambda}.$$

As $\lambda \rightarrow 0$ the formula tends to x , so the 2 cases join continuously.

Exercise 1.3. (a) The volume changes at the rate $V'(t) = q(t) - kV(t)$. Integrating from 0 to x and using $V(0) = V_0$ gives

$$V(x) = V_0 + \int_0^x q(t) \, dt - k \int_0^x V(t) \, dt.$$

Setting $x = 0$ recovers $V(0) = V_0$, so the initial condition is built in.

(b) With $q = q_0$ the equation is $V(x) = V_0 + q_0 x - k \int_0^x V(t) \, dt$. Differentiating gives $V' = q_0 - kV$ with $V(0) = V_0$. The constant q_0/k solves the differential equation, and the difference $V - q_0/k$ satisfies $w' = -kw$, so

$$V(x) = \frac{q_0}{k} + \left(V_0 - \frac{q_0}{k} \right) e^{-kx}.$$

Check in the integral equation: $\int_0^x V(t) \, dt = q_0 x/k + (V_0 - q_0/k)(1 - e^{-kx})/k$, so $V_0 + q_0 x - k \int_0^x V = V_0 + q_0 x - q_0 x - (V_0 - q_0/k)(1 - e^{-kx}) = q_0/k + (V_0 - q_0/k)e^{-kx}$, as required. As $x \rightarrow \infty$ the volume approaches q_0/k , the level at which drainage exactly balances inflow.

Exercise 1.4. Differentiating an equation keeps only the information about derivatives; the value of both sides at one point is lost. The integral equation carries that value: at $x = 0$ the integral vanishes, so $u(0) = 1$. Among the functions $Ce^{\lambda x}$ only $C = 1$ satisfies this.

Substituting $u = Ce^{\lambda x}$ with $\lambda \neq 0$ into the original equation shows the failure directly: the right-hand side is $1 + C(e^{\lambda x} - 1) = Ce^{\lambda x} + (1 - C)$, which differs from the left-hand side by the constant $1 - C$. So the integral equation has exactly one continuous solution, $e^{\lambda x}$. The general lesson: any step that differentiates must be paired with the condition that recovers what differentiation threw away, and the final answer must be checked in the original equation.

Exercise 1.5. (a) The unknown is u , the data are $f(x) = \sin x$, the kernel is $K(x, t) = \cos(x + t)$ and the parameter is $\lambda = 1$. The limits are fixed at 0 and π , and u appears outside the integral as well as inside.

(b) The unknown is w , the data are x^2 and the kernel is e^{x-t} . There is no parameter in front of the integral (or $\lambda = 1$ if you prefer to insert one). The upper limit is the variable x , and w appears only inside the integral. This is the inverse kind of question from (1.9).

(c) The letters have changed but the structure has not: the unknown is ϕ , the free variable is s and the integration variable is r . The data are the constant 1, the kernel is $K(s, r) = (s-r)^2$, the parameter is $\frac{1}{2}$ and the limits are -1 and 1 . The unknown appears outside the integral too.

Exercise 1.6. Split the integral at $t = x$:

$$u(x) = (1-x) \int_0^x t \, dt + x \int_x^1 (1-t) \, dt = (1-x) \frac{x^2}{2} + x \frac{(1-x)^2}{2} = \frac{x(1-x)}{2}.$$

Then $u''(x) = -1$, so $-u'' = 1$, and $u(0) = u(1) = 0$. The deflection is a parabola with its greatest depth $1/8$ at the middle, which matches the picture of a string under its own uniform weight.

Exercise 1.7. Splitting at $t = x$ and integrating by parts,

$$\begin{aligned} \int_0^x t \sin(2\pi t) \, dt &= -\frac{x \cos(2\pi x)}{2\pi} + \frac{\sin(2\pi x)}{4\pi^2}, \\ \int_x^1 (1-t) \sin(2\pi t) \, dt &= \frac{(1-x) \cos(2\pi x)}{2\pi} + \frac{\sin(2\pi x)}{4\pi^2}. \end{aligned}$$

Multiplying the first by $1 - x$, the second by x and adding, the cosine terms cancel and $\int_0^1 G(x, t) \sin(2\pi t) dt = \sin(2\pi x)/(4\pi^2)$. With $u = A \sin(2\pi x)$, equation (1.5) becomes $A = 1/(4\pi^2) - kA/(4\pi^2)$, so $A = 1/(4\pi^2 + k)$.

Without the foundation the amplitude is $1/(4\pi^2)$, so the foundation multiplies it by $4\pi^2/(4\pi^2 + k)$. For the load $\sin(\pi x)$ the factor was $\pi^2/(\pi^2 + k)$. The string is already 4 times stiffer against the more wiggly load, so the same springs make a smaller relative difference.

Exercise 1.8. At $x = 0$ the integral vanishes, so $B(0) = 1$. Differentiating once with the Leibniz rule (the integrand $(x - t)B(t)$ vanishes at $t = x$) gives $B'(x) = \beta \int_0^x B(t) dt$, so $B'(0) = 0$. Differentiating again gives $B'' = \beta B$. The solution with $B(0) = 1$ and $B'(0) = 0$ is $B(x) = \cosh(\sqrt{\beta} x)$.

Check: $\int_0^x (x - t) \cosh(\sqrt{\beta} t) dt = (\cosh(\sqrt{\beta} x) - 1)/\beta$ by integrating by parts twice, so $1 + \beta \int_0^x (x - t)B(t) dt = \cosh(\sqrt{\beta} x)$. The births grow like $e^{\sqrt{\beta} x}/2$ for large x .

Exercise 1.9. For $n = 0$ the sum is $1 = u_0(x)$. Suppose the formula holds for some $n \geq 0$. Then

$$u_{n+1}(x) = 1 + \lambda \int_0^x \sum_{k=0}^n \frac{(\lambda t)^k}{k!} dt = 1 + \sum_{k=0}^n \frac{\lambda^{k+1}}{k!} \cdot \frac{x^{k+1}}{k+1} = 1 + \sum_{k=1}^{n+1} \frac{(\lambda x)^k}{k!} = \sum_{k=0}^{n+1} \frac{(\lambda x)^k}{k!},$$

which is the formula for $n + 1$. Exchanging the finite sum and the integral needs no justification. By induction the formula holds for every n .

The Taylor series $\sum_{k \geq 0} z^k/k!$ converges to e^z for every real z , so $u_n(x) \rightarrow e^{\lambda x}$ for every x . Iteration therefore rediscovers the solution of Example 1.1 without any differentiation.

Exercise 1.10. (a) The integral $\int_0^1 u(t) dt$ is a single number, so any solution must be a constant function $u(x) = c$. Substituting gives $c = 1 + \int_0^1 c dt = 1 + c$, that is, $0 = 1$. No constant, and therefore no function, solves the equation.

(b) Again any solution is a constant c , and now substitution gives $c = c$. Every constant is a solution, so there are infinitely many.

In Benchmark B1 the integral runs from 0 to x , so the value of u at x depends only on values at earlier points, and the equation can be solved by marching forward from $u(0) = 1$. With the fixed upper limit, every value depends on every other value at once, and for this kernel the parameter $\lambda = 1$ is exactly the value at which the equation loses either existence or uniqueness. Chapters 5 and 12 make this precise.

Exercise 1.11. The partial sums are $s_1 = 2$, $s_2 = 2.5$, $s_3 = 2.6667$, $s_4 = 2.70833$, $s_5 = 2.71667$ and $s_6 = 2.718056$, so with $e = 2.718282$ the errors are

$$0.7183, \quad 0.2183, \quad 0.05162, \quad 0.009948, \quad 0.001615, \quad 0.0002263.$$

The first omitted terms $1/(n+1)!$ are 0.5, 0.1667, 0.04167, 0.008333, 0.001389 and 0.0001984. The ratio of error to first omitted term is 1.44, 1.31, 1.24, 1.19, 1.16 and 1.14, decreasing towards 1. The tail of the series is $\frac{1}{(n+1)!}(1 + \frac{1}{n+2} + \dots)$, and the bracket is below $1 + \frac{1}{n+1}$, which explains the pattern. The factorial in the denominator makes the error fall faster than any geometric rate, a feature of Volterra equations that Chapter 7 explains.

Exercise 1.12. Try $B(x) = B_0$. Then $\rho \int_0^1 F(x, t) B_0 dt = \rho \varphi B_0$ for every x , and the equation becomes $B_0 = E_0 + \rho \varphi B_0$. Since $\rho < 1$ and $\varphi \leq 1$, the number $1 - \rho \varphi$ is positive, and $B_0 = E_0/(1 - \rho \varphi)$.

The same number appears if you follow the light: the wall emits E_0 , a fraction $\rho \varphi$ of that returns after one bounce, a fraction $(\rho \varphi)^2$ after 2 bounces, and so on. The total is $E_0 \sum_{n \geq 0} (\rho \varphi)^n = E_0/(1 - \rho \varphi)$. A closed enclosure, in which every ray leaving the wall lands on the wall again ($\varphi = 1$), with walls that reflect almost everything (ρ near 1) is many times brighter than its own emission alone.

Exercise 1.13. Subtracting the 2 equations removes the forcing term, so $w(x) = \lambda \int_0^x w(t) dt$ on $[0, X]$. The function w is continuous on a closed interval, so $M = \max|w|$ is finite, and the bound holds for $n = 0$.

Suppose $|w(t)| \leq M|\lambda|^n t^n/n!$ for all $t \in [0, X]$. Then for $x \in [0, X]$,

$$|w(x)| \leq |\lambda| \int_0^x |w(t)| dt \leq |\lambda| \int_0^x \frac{M|\lambda|^n t^n}{n!} dt = \frac{M|\lambda|^{n+1} x^{n+1}}{(n+1)!},$$

so the bound holds for $n + 1$. By induction it holds for all n .

For fixed x , the number $(|\lambda|x)^n/n!$ is the n th term of the convergent series for $e^{|\lambda|x}$, so it tends to 0. Hence $|w(x)| \leq 0$, and $w(x) = 0$ for every $x \in [0, X]$. The 2 solutions coincide.

The argument never used a derivative. It used only the triangular shape of the integration region, which produces the factorial, and that same estimate drives the general theory of Volterra equations in Chapter 7.

Chapter 2 Types of Equations and Kernels

Exercise 2.1. (a) The limits are fixed and the kernel is nonzero everywhere, so the equation is Fredholm; the unknown appears outside with coefficient 1, so it is second kind; it is linear and nonhomogeneous ($f = x$). The kernel $e^{x+t} = e^x e^t$ is separable of rank 1 and symmetric.

(b) The upper limit is x and the unknown appears only inside: a first-kind Volterra equation, linear and nonhomogeneous. The kernel depends on $x - t$ only, so it is a convolution kernel; it is also separable, $\sin x \cos t - \cos x \sin t$, of rank 2.

(c) A homogeneous second-kind Fredholm equation with the symmetric rank-one kernel $\sin x \sin t$. Its only characteristic value is $\lambda = 2/\pi$.

(d) A second-kind Volterra equation. The unknown enters the integrand as u^2 , so it is nonlinear, of Hammerstein form with $K(x, t) = t$ and $g(t, u) = u^2$.

(e) The coefficient $1 - x^2$ vanishes at $x = \pm 1$, the ends of the interval, so this is a third-kind Fredholm equation. It is linear and nonhomogeneous, with the constant kernel 1 (separable of rank 1, symmetric).

(f) A second-kind Fredholm equation, linear and nonhomogeneous. The kernel $|x - t|^{-1/3}$ is weakly singular with $\alpha = 1/3$ and symmetric, and it depends only on $|x - t|$.

Exercise 2.2. Define $\tilde{K}(x, t) = e^{x-t}$ when $0 \leq t \leq x \leq 1$ and $\tilde{K}(x, t) = 0$ when $t > x$. Then $\int_0^1 \tilde{K}(x, t)u(t) dt = \int_0^x e^{x-t}u(t) dt$, so the equation is $u(x) = x + \int_0^1 \tilde{K}(x, t)u(t) dt$.

Approaching a diagonal point (x_0, x_0) from below ($t < x$), $\tilde{K} \rightarrow e^0 = 1$; from above, $\tilde{K} = 0$. So $\tilde{K}$ is discontinuous along the whole diagonal.

In general, if K is continuous on the closed triangle, the extension by zero is continuous on the square if and only if its values on the diagonal from below match the zero values from above, that is, $K(x, x) = 0$ for every x . The kernel $(x - t)$ of Example 2.1 is an example.

Exercise 2.3. (a) Here $K(x, t) = 1 + x - t$, so $K(x, x) = 1$ and $\partial_x K = 1$, and $f(x) = x$ satisfies $f(0) = 0$. By Proposition 2.3 the equation is equivalent to

$$u(x) = 1 - \int_0^x u(t) dt.$$

Setting $x = 0$ gives $u(0) = 1$, and differentiating gives $u' = -u$, so $u(x) = e^{-x}$. Check in the original equation:

$$\int_0^x e^{-t} dt + \int_0^x (x-t)e^{-t} dt = (1 - e^{-x}) + (x - 1 + e^{-x}) = x.$$

(b) Now $K(x, x) = 0$. Differentiating once with (2.4) gives $\int_0^x u(t) dt = 2x$, which is still first kind, now with kernel 1 and $K(x, x) = 1$. Differentiating again gives $u(x) = 2$. Going back up requires 2 conditions on the data, $f(0) = 0$ and $f'(0) = 0$, and $f(x) = x^2$ satisfies both. Check: $\int_0^x (x-t) \cdot 2 dt = x^2$.

Exercise 2.4. The rank of $\sum_{i=1}^n a_i(x)b_i(t)$ equals n when the functions $a_1, \dots, a_n$ are linearly independent and so are $b_1, \dots, b_n$; otherwise terms can be merged.

(a) $\sin(x+t) = \sin x \cos t + \cos x \sin t$. The pairs $\{\sin x, \cos x\}$ and $\{\cos t, \sin t\}$ are independent, so the rank is 2.

(b) $e^{x+t} = e^x \cdot e^t$ has rank 1.

(c) $(x-t)^2 = x^2 \cdot 1 + x \cdot (-2t) + 1 \cdot t^2$. The functions $x^2, x, 1$ are independent and so are $1, -2t, t^2$, so the rank is 3.

(d) $1 + \cos(x-t) = 1 \cdot 1 + \cos x \cos t + \sin x \sin t$. On $[0, \pi]$ the functions $1, \cos x, \sin x$ are independent, so the rank is 3.

Exercise 2.5. (a) $e^{-|t-x|} = e^{-|x-t|}$, and the kernel is real, so it is symmetric and Hermitian.

(b) $K(t, x) = tx^2 \neq xt^2$ in general; at $x = 1, t = \frac{1}{2}$ the values are $K(x, t) = \frac{1}{4}$ and $K(t, x) = \frac{1}{2}$. It is real and not symmetric, so it is neither.

(c) $K(t, x) = i(t-x) = -K(x, t)$, so it is not symmetric; $\overline{K(t, x)} = -i(t-x) = i(x-t) = K(x, t)$, so it is Hermitian.

(d) $K(t, x) = e^{i(t-x)} \neq K(x, t)$ unless $x-t$ is a multiple of π , so it is not symmetric; $\overline{K(t, x)} = e^{-i(t-x)} = e^{i(x-t)} = K(x, t)$, so it is Hermitian.

(e) $K(t, x) = e^{i(t+x)} = K(x, t)$, so it is symmetric; $\overline{K(t, x)} = e^{-i(x+t)} \neq K(x, t)$ in general, so it is not Hermitian. This is the case the text warns about: complex symmetry without the conjugate does not give a real spectrum.

Exercise 2.6. Write $T\varphi = xA + B$ with $A = \int_0^1 \varphi$ and $B = \int_0^1 t\varphi$. An eigenfunction with $\mu \neq 0$ has the form $\varphi = (Ax + B)/\mu$, and substituting into the definitions of A and B gives

$$\mu A = \frac{1}{2}A + B, \quad \mu B = \frac{1}{3}A + \frac{1}{2}B.$$

The matrix $\begin{pmatrix} 1/2 & 1 \\ 1/3 & 1/2 \end{pmatrix}$ has eigenvalues $\mu = \frac{1}{2} \pm \frac{1}{\sqrt{3}}$. The first equation gives $B = (\mu - \frac{1}{2})A = \pm A/\sqrt{3}$, so the eigenfunctions are $x \pm 1/\sqrt{3}$.

The characteristic values are the reciprocals:

$$\lambda = \frac{1}{\frac{1}{2} \pm \frac{1}{\sqrt{3}}} = \frac{2\sqrt{3}}{\sqrt{3} \pm 2} = -6 \pm 4\sqrt{3}.$$

Numerically, $\lambda_1 = 4\sqrt{3} - 6 \approx 0.928$ with eigenfunction $x + 1/\sqrt{3}$, and $\lambda_2 = -4\sqrt{3} - 6 \approx -12.93$ with eigenfunction $x - 1/\sqrt{3}$. Both are real, as they must be for a real symmetric kernel.

Exercise 2.7. Substituting $\lambda = 0$ into $u = \lambda Tu$ gives $u = 0 \cdot Tu = 0$. The only solution is the zero function, which by definition is not an eigenfunction. So $\lambda = 0$ is never a characteristic value, for any kernel. The notes confused the equation $u = \lambda Tu$ with the equation $Tu = \mu u$.

What can be special about 0 is the operator eigenvalue $\mu = 0$: nonzero functions with $T\varphi = 0$. For $K(x, t) = xt$ on $[0, 1]$, $T\varphi = x \int_0^1 t\varphi(t) dt$, which vanishes exactly when $\int_0^1 t\varphi(t) dt = 0$. For example $\varphi(x) = 2 - 3x$ gives $\int_0^1 (2t - 3t^2) dt = 1 - 1 = 0$, and $3x - 4x^2$ gives $\int_0^1 (3t^2 - 4t^3) dt = 1 - 1 = 0$; infinitely many other functions work too. These make up the null space of T . Because $\lambda = 1/\mu$ has no meaning for $\mu = 0$, the null space corresponds to no characteristic value. The only characteristic value of this kernel is $\lambda = 3$.

Exercise 2.8. Write $\sin(x - t) = \sin x \cos t - \cos x \sin t$ and let $A = \int_0^\pi \cos t \varphi(t) dt$, $B = \int_0^\pi \sin t \varphi(t) dt$. Then $T\varphi = A \sin x - B \cos x$, and an eigenfunction with $\mu \neq 0$ is $\varphi = (A \sin x - B \cos x)/\mu$. Using $\int_0^\pi \sin t \cos t dt = 0$ and $\int_0^\pi \cos^2 t dt = \int_0^\pi \sin^2 t dt = \pi/2$,

$$\mu A = -\frac{\pi}{2}B, \quad \mu B = \frac{\pi}{2}A.$$

Multiplying the first by μ and using the second gives $\mu^2 A = -\frac{\pi^2}{4}A$. If $A = 0$ then $B = 0$ too and $\varphi = 0$, so $A \neq 0$ and $\mu^2 = -\pi^2/4$. Hence $\mu = \pm i\pi/2$ and $\lambda = 1/\mu = \mp 2i/\pi$. No real λ gives a nonzero solution, so for real parameters the equation $u = f + \lambda Tu$ always has at most one solution.

Exercise 2.9. Split the integral at $t = x$ and treat each piece as an improper integral:

$$\int_0^x (x-t)^{-\alpha} dt = \left[-\frac{(x-t)^{1-\alpha}}{1-\alpha} \right]_0^x = \frac{x^{1-\alpha}}{1-\alpha}, \quad \int_x^1 (t-x)^{-\alpha} dt = \frac{(1-x)^{1-\alpha}}{1-\alpha},$$

where the evaluation at $t = x$ gives 0 because $1 - \alpha > 0$. Adding gives the formula. For $\alpha = \frac{1}{2}$ it is $2(\sqrt{x} + \sqrt{1-x})$; at $x = \frac{1}{2}$ this is $2\sqrt{2}$, the finite area in Figure 2.2.

For $\alpha = 1$, $\int_0^{x-\varepsilon} (x-t)^{-1} dt = \log x - \log \varepsilon$, which tends to $+\infty$ as $\varepsilon \rightarrow 0^+$. The singularity is not integrable, which is why Cauchy kernels need principal values.

Exercise 2.10. (a) At $x = 0$ the left-hand side is 0 and the integral term is $0 \cdot \int_0^1 t u(t) dt = 0$, so the equation forces $f(0) = 0$.

(b) Let $c = \int_0^1 t u(t) dt$. For $x \neq 0$, dividing by x gives $u(x) = x + \lambda c$, and a continuous solution must have this form on all of $[0, 1]$. Then $c = \int_0^1 t(t + \lambda c) dt = \frac{1}{3} + \frac{\lambda c}{2}$, so $(1 - \frac{\lambda}{2})c = \frac{1}{3}$. For $\lambda \neq 2$, $c = \frac{2}{3(2-\lambda)}$ and $u(x) = x + \frac{2\lambda}{3(2-\lambda)}$. Check: $x u(x) = x^2 + \lambda x c$, which is the right-hand side. For $\lambda = 2$ the coefficient equation reads $0 = \frac{1}{3}$, so there is no solution.

(c) With $f = 1$, part (a) fails: at $x = 0$ the equation reads $0 = 1$. No continuous (indeed no finite) value of $u(0)$ helps, because $u(0)$ is multiplied by 0. For $x > 0$ the formula $u = 1/x + \lambda c$ is forced, and it is unbounded near 0.

Exercise 2.11. For $u_p = x - \frac{2}{3}$: $\int_0^1 t(t - \frac{2}{3}) dt = \frac{1}{3} - \frac{1}{3} = 0$, so the right-hand side is $x - \frac{2}{3} + 0 = u_p$. For $\varphi = x$: $3x \int_0^1 t \cdot t dt = 3x \cdot \frac{1}{3} = x = \varphi$. By Proposition 2.4, any solution u differs from u_p by a solution w of the homogeneous equation. Every solution of $w = 3x \int_0^1 t w$ is a multiple of x (the right-hand side is), and every multiple works. So the solutions are exactly $u = x - \frac{2}{3} + Cx$, $C \in \mathbb{R}$, the infinite family listed with the benchmark.

Exercise 2.12. By the benchmark formula $u = a/(1 - a\lambda x)$, the solution with $a = 2$, $\lambda = 1$ is $2/(1 - 2x)$. Test $v = 2/(1 - x)$ directly: $2 + \int_0^x v(t)^2 dt = 2 + 4(\frac{1}{1-x} - 1) = \frac{4}{1-x} - 2$, which is not v (at $x = \frac{1}{2}$ it is 6, while $v = 4$). So doubling the data does not double the solution, and superposition fails. The pole also moves: doubling a halves the interval of existence, from $[0, 1]$ to $[0, \frac{1}{2}]$. A linear equation could not do this, because scaling the data of a linear equation scales the solution without changing where it exists.

Exercise 2.13. Let $\lambda \in \mathbb{C}$ and let $\varphi \neq 0$ be a continuous complex-valued function with $\varphi = \lambda T\varphi$. Taking the inner product with φ ,

$$\langle \varphi, \varphi \rangle = \lambda \langle T\varphi, \varphi \rangle, \quad \langle T\varphi, \varphi \rangle = \int_a^b \int_a^b K(x, t) \varphi(t) \overline{\varphi(x)} dt dx.$$

The integrand is continuous on the square, so the double integral may be computed in either order and its variables renamed. Taking the complex conjugate, and using that K is real,

$$\overline{\langle T\varphi, \varphi \rangle} = \int_a^b \int_a^b K(x, t) \overline{\varphi(t)} \varphi(x) dt dx = \int_a^b \int_a^b K(t, x) \overline{\varphi(x)} \varphi(t) dx dt,$$

where the second step exchanges the names x and t . By symmetry $K(t, x) = K(x, t)$, so the last integral is $\langle T\varphi, \varphi \rangle$ again. Hence $\langle T\varphi, \varphi \rangle$ is real.

Since φ is continuous and not identically zero, $\langle \varphi, \varphi \rangle = \int_a^b |\varphi|^2 > 0$. The identity $\langle \varphi, \varphi \rangle = \lambda \langle T\varphi, \varphi \rangle$ then shows $\langle T\varphi, \varphi \rangle \neq 0$, and

$$\lambda = \frac{\langle \varphi, \varphi \rangle}{\langle T\varphi, \varphi \rangle}$$

is a quotient of 2 real numbers, so it is real. The same proof works for a complex Hermitian kernel, with $K(t, x) = \overline{K(x, t)}$ used in place of symmetry.

Exercise 2.14. A numerical eigenvalue routine applied to the 100 by 100 matrix returns 98 eigenvalues that are zero to rounding error and the pair $\pm 0.28866070 i$. The exact values are $\pm 1/(2\sqrt{3}) i = \pm 0.28867513 i$.

The error can be found exactly. The matrix has rank 2, and the same reduction as in Example 2.5 applies with the integrals replaced by midpoint sums $S_0 = \sum_j 1/N = 1$, $S_1 = \sum_j t_j/N = \frac{1}{2}$ and $S_2 = \sum_j t_j^2/N = \frac{1}{3} - \frac{1}{12N^2}$. The nonzero eigenvalues satisfy $\mu^2 = S_1^2 - S_0 S_2 = -\frac{1}{12} + \frac{1}{12N^2}$, so $\mu = \pm i\sqrt{(1 - N^{-2})/12}$. The relative error is $1 - \sqrt{1 - N^{-2}} \approx 1/(2N^2)$, which is 5×10^{-5} for $N = 100$. The error comes entirely from the midpoint rule's error in $\int_0^1 t^2 dt$, a pattern Chapter 18 makes general.

Chapter 3 The Mathematical Toolkit

Exercise 3.1. The region is $\{0 \leq x \leq t \leq 1\}$. For fixed t , x runs from 0 to t , so

$$\int_0^1 \int_x^1 e^{t^2} dt dx = \int_0^1 e^{t^2} \left(\int_0^t dx \right) dt = \int_0^1 t e^{t^2} dt = \left[\frac{1}{2} e^{t^2} \right]_0^1 = \frac{e-1}{2}.$$

The original order cannot be done in elementary terms because e^{t^2} has no elementary antiderivative; the swap removes the difficulty.

Exercise 3.2. The inner double integral is $\int_0^s \int_0^r g(t) dt dr$. Dirichlet's formula on the triangle $0 \leq t \leq r \leq s$ gives $\int_0^s g(t) \int_t^s dr dt = \int_0^s (s-t)g(t) dt$. The remaining integral is

$$\int_0^x \int_0^s (s-t)g(t) dt ds = \int_0^x g(t) \int_t^x (s-t) ds dt = \int_0^x \frac{(x-t)^2}{2} g(t) dt,$$

again by Dirichlet's formula, now on $0 \leq t \leq s \leq x$ with $F(s, t) = (s-t)g(t)$.

Check with $g(t) = t$: the repeated integral is $\int_0^x \int_0^s r^2/2 dr ds = \int_0^x s^3/6 ds = x^4/24$, and $\int_0^x \frac{1}{2}(x-t)^2 t dt = \frac{1}{2}(x^4/2 - 2x^4/3 + x^4/4) = x^4/24$.

Exercise 3.3. Expand the product: $(x+s)(s+t) = xt + (x+t)s + s^2$. Integrating over $s \in [0, 1]$ gives $xt + \frac{1}{2}(x+t) + \frac{1}{3}$. The swap that makes K_2 the kernel of $T(Tu)$ is justified by Theorem 3.1, since the integrand $K(x, s)K(s, t)u(t)$ is continuous on the cube for continuous u .

Exercise 3.4. For $t \leq s \leq x$, $K(x, s)K(s, t) = e^{x-s}e^{s-t} = e^{x-t}$ does not depend on s . Integrating over s from t to x multiplies it by $x-t$, so the composed kernel is $(x-t)e^{x-t}$. The limits t and x come from the triangle, as in Example 3.2.

Exercise 3.5. The integrand $F(x, t) = \sin(x-t)g(t)$ and $\partial_x F = \cos(x-t)g(t)$ are continuous on the triangle, so (3.6) applies:

$$v'(x) = \sin(0)g(x) + \int_0^x \cos(x-t)g(t) dt = \int_0^x \cos(x-t)g(t) dt.$$

Apply the rule again: $v''(x) = \cos(0)g(x) - \int_0^x \sin(x-t)g(t) dt = g(x) - v(x)$. Setting $x = 0$ in the formulas for v and v' gives $v(0) = v'(0) = 0$ because the intervals of integration have length zero.

Exercise 3.6. By (3.5) with $\alpha = 0$, $\beta = x^2$ and $F = \cos(xt)$,

$$\frac{d}{dx} \int_0^{x^2} \cos(xt) dt = \cos(x^3) \cdot 2x - \int_0^{x^2} t \sin(xt) dt.$$

The quoted derivation dropped both the chain-rule factor $2x$ and the integral of $\partial F/\partial x$. To check, for $x \neq 0$ compute $\int_0^{x^2} \cos(xt) dt = \sin(x^3)/x$, whose derivative is $3x \cos(x^3) - \sin(x^3)/x^2$. Integration by parts gives $\int_0^{x^2} t \sin(xt) dt = -x \cos(x^3) + \sin(x^3)/x^2$, so the Leibniz expression equals $2x \cos(x^3) + x \cos(x^3) - \sin(x^3)/x^2$, the same function.

Exercise 3.7. At $t = 0$ and $t = 1$, $g_n(t) = 0$. For $0 < t < 1$, $0 < 1 - t^2 < 1$ and $n(1 - t^2)^n \rightarrow 0$ because geometric decay beats linear growth. With $w = 1 - t^2$,

$$\int_0^1 nt(1 - t^2)^n dt = \frac{n}{2} \int_0^1 w^n dw = \frac{n}{2(n+1)} \rightarrow \frac{1}{2}.$$

The maximum of g_n occurs where $1 - t^2 = 2nt^2$, at $t_n = 1/\sqrt{2n+1}$, and $g_n(t_n) = \frac{n}{\sqrt{2n+1}} \left(\frac{2n}{2n+1}\right)^n \approx \frac{n}{\sqrt{2n}} e^{-1/2}$, which grows without bound. So $\sup|g_n - 0| \not\rightarrow 0$ and Theorem 3.7 does not apply.

Exercise 3.8. On the triangle, $0 \leq x - t \leq b$, so $|\lambda^n(x - t)^{2n}/(2n)!| \leq (|\lambda|b^2)^n/(2n)! = M_n$, and $\sum M_n = \cosh(\sqrt{|\lambda|}b) < \infty$. The M-test gives absolute and uniform convergence. For $\lambda = \omega^2 > 0$ the series is $\sum(\omega(x - t))^{2n}/(2n)! = \cosh(\omega(x - t))$; for $\lambda = -\omega^2 < 0$ it is $\sum(-1)^n(\omega(x - t))^{2n}/(2n)! = \cos(\omega(x - t))$.

Exercise 3.9. Suppose such a constant C existed. The functions of Example 3.7 have $\|u_n\|_\infty = 1$ and $\|u_n\|_2 = 1/\sqrt{2n+1}$, so $1 \leq C/\sqrt{2n+1}$, that is $2n+1 \leq C^2$, for every n . This is false for $n > C^2/2$. Hence mean-square smallness never guarantees pointwise smallness on its own.

Exercise 3.10. By Theorem 3.11,

$$|(Tu)(x)| \leq \left(\int_0^1 e^{2xt} dt \right)^{1/2} \|u\|_2 = \sqrt{(e^{2x} - 1)/(2x)} \|u\|_2.$$

Equality in Cauchy–Schwarz holds when the two functions are proportional; with $u(t) = e^{x_0 t}$ we get $(Tu)(x_0) = \int_0^1 e^{2x_0 t} dt = (e^{2x_0} - 1)/(2x_0)$, and $\|u\|_2 = \sqrt{(e^{2x_0} - 1)/(2x_0)}$, so both sides equal $(e^{2x_0} - 1)/(2x_0)$.

Exercise 3.11. For $m \neq n$, $\cos(m\pi x) \cos(n\pi x) = \frac{1}{2}[\cos((m-n)\pi x) + \cos((m+n)\pi x)]$, and each cosine of a nonzero integer multiple of πx integrates to 0 over $[0, 1]$. The norms are $\|1\|_2^2 = 1$ and $\|\cos(n\pi x)\|_2^2 = \frac{1}{2}$ for $n \geq 1$. By Theorem 3.12 applied to the orthonormal pair 1 and $\sqrt{2} \cos(\pi x)$,

$$c_0 = \int_0^1 x \, dx = \frac{1}{2}, \quad c_1 = 2 \int_0^1 x \cos(\pi x) \, dx = 2 \cdot \frac{\cos \pi - 1}{\pi^2} = -\frac{4}{\pi^2}.$$

Exercise 3.12. The eigenvalues of A solve $(2 - \mu)^2 = 1$, so $\mu = 1$ or $\mu = 3$, with eigenvectors $(1, -1)$ and $(1, 1)$. By the discussion after Theorem 3.13, $\det(I - \lambda A) = 0$ exactly for $\lambda = 1/\mu$, that is $\lambda = 1$ and $\lambda = \frac{1}{3}$.

At $\lambda = 1$, $M = I - A = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$ is symmetric with null space spanned by $(1, -1)$. By Theorem 3.13, $M\mathbf{c} = \mathbf{b}$ is solvable if and only if $\mathbf{b} \cdot (1, -1) = 0$, that is $b_1 = b_2$. For $\mathbf{b} = (1, 1)$ both equations read $-c_1 - c_2 = 1$, so $c_1 + c_2 = -1$. One solution is $(-\frac{1}{2}, -\frac{1}{2})$, and adding any multiple of the null vector gives all of them: $\mathbf{c} = (-\frac{1}{2}, -\frac{1}{2}) + C(1, -1)$. Check: $M(-\frac{1}{2}, -\frac{1}{2}) = (1, 1)$ and $M(1, -1) = (0, 0)$.

Exercise 3.13. Integration by parts gives $\int_0^x (x-t)e^t \, dt = [(x-t)e^t]_0^x + \int_0^x e^t \, dt = -x + e^x - 1$. Its transform is $\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} = \frac{s^2 - s(s-1) - (s-1)}{s^2(s-1)} = \frac{1}{s^2(s-1)}$ for $s > 1$. The convolution theorem predicts $\mathcal{L}\{t\}\mathcal{L}\{e^t\} = \frac{1}{s^2} \cdot \frac{1}{s-1}$, the same function.

Exercise 3.14. The kernel $k(y) = y^{n-1}/(n-1)!$ is of exponential order γ for every $\gamma > 0$ and has $\mathcal{L}\{k\} = 1/s^n$ by Table 3.2. Theorem 3.16 gives the transform $G(s)/s^n$ for s larger than both 0 and the exponential order of g . For $n = 1$ this is the rule $\mathcal{L}\{\int_0^x g\} = G/s$, which also follows from Theorem 3.14 applied to $f = \int_0^x g$ with $f(0) = 0$. Applying it n times to the repeated integral $I^n g$ gives G/s^n , consistent with Corollary 3.3 and the uniqueness of the transform.

Exercise 3.15. With $t = x - r$, $\Phi(x) = \int_0^x r^{-1/2} g(x-r) \, dr$. The singular factor $r^{-1/2}$ no longer depends on x . For $0 < \eta < x$ split $\Phi(x) = \int_0^\eta r^{-1/2} g(x-r) \, dr + \int_\eta^x r^{-1/2} g(x-r) \, dr$. On $[\eta, x]$ the integrand and its x -derivative are continuous, so the Leibniz rule gives the derivative $x^{-1/2} g(0) + \int_\eta^x r^{-1/2} g'(x-r) \, dr$ for the second piece. The first piece has fixed limits and x -derivative $\int_0^\eta r^{-1/2} g'(x-r) \, dr$; differentiating under the integral is justified because $|r^{-1/2} g'(x-r)| \leq \|g'\|_\infty r^{-1/2}$, an integrable bound independent of x (dominated convergence [56]). Adding the pieces,

$$\Phi'(x) = g(0)x^{-1/2} + \int_0^x r^{-1/2} g'(x-r) \, dr = g(0)x^{-1/2} + \int_0^x (x-t)^{-1/2} g'(t) \, dt.$$

For $g(t) = t$: $\Phi(x) = \int_0^x (x-t)^{-1/2} t \, dt = \frac{4}{3} x^{3/2}$, so $\Phi'(x) = 2x^{1/2}$. The formula gives $0 + \int_0^x (x-t)^{-1/2} \, dt = 2x^{1/2}$.

Exercise 3.16. Since $e^{xs}e^{st} = e^{(x+t)s}$, the integral is $\left[e^{(x+t)s}/(x+t) \right]_0^1 = (e^{x+t}-1)/(x+t)$. At $(0.3, 0.7)$ this is $e-1 \approx 1.7182818$. The midpoint rule with m subintervals of width $h = 1/m$ gives $h \sum_{j=1}^m e^{(j-1/2)h}$, which equals 1.7175660, 1.7181028 and 1.7182371 for $m = 10, 20, 40$. The errors -7.16×10^{-4} , -1.79×10^{-4} and -4.47×10^{-5} fall by a factor close to 4 at each halving, the second-order behavior of the midpoint rule that Chapter 18 explains.

Chapter 4 Verification and Method Selection

Exercise 4.1. Integration by parts gives $\int_0^x (x-t) \cos t \, dt = 1 - \cos x$ and $\int_0^x (x-t) \sin t \, dt = x - \sin x$. For $u_c = \cos x$ the right side is $1 - (1 - \cos x) = \cos x$, so the residual $u_c - 1 + \int_0^x (x-t) u_c \, dt$ is 0. For $u_c = \sin x + \cos x$ the residual is $\sin x + \cos x - 1 + (x - \sin x) + (1 - \cos x) = x$. The candidate violates the hidden condition $u'(0) = 0$.

Exercise 4.2. Transform: $\mathcal{L}\{\cos\} = s/(s^2 + 1)$, so $U = 1/s^2 + 2sU/(s^2 + 1)$ and $U(s - 1)^2/(s^2 + 1) = 1/s^2$, giving $U = (s^2 + 1)/(s^2(s - 1)^2)$. Partial fractions give $2/s + 1/s^2 - 2/(s - 1) + 2/(s - 1)^2$, whose inverse transform is $2 + x - 2e^x + 2xe^x = x + 2 + 2(x - 1)e^x$.

For the substitution, compute the convolutions $\int_0^x \cos(x-t) \, dt = \sin x$, $\int_0^x \cos(x-t) t \, dt = 1 - \cos x$, $\int_0^x \cos(x-t) e^t \, dt = \frac{1}{2}(e^x - \cos x + \sin x)$ and $\int_0^x \cos(x-t) t e^t \, dt = \frac{1}{2}(xe^x - \sin x)$. Then $\int_0^x \cos(x-t) u(t) \, dt = 2 \sin x + (1 - \cos x) + (xe^x - \sin x) - (e^x - \cos x + \sin x) = 1 + xe^x - e^x$, and $x + 2(1 + xe^x - e^x) = x + 2 + 2(x - 1)e^x = u(x)$.

Exercise 4.3. The residual is $u_c - 1 + \int_0^x u_c(t) \, dt = (1 - x + x^2/2) - 1 + (x - x^2/2 + x^3/6) = x^3/6$, with maximum $1/6$ at $x = 1$. Let $g(x) = u_c(x) - e^{-x} = 1 - x + x^2/2 - e^{-x}$, so $e = -g$. Then $g(0) = 0$ and $g'(x) = -1 + x + e^{-x} \geq 0$ because $e^{-x} \geq 1 - x$. So g increases, and $\|e\|_\infty = g(1) = \frac{1}{2} - e^{-1} \approx 0.1321$. Here the error is smaller than the residual, because the solution decays.

Exercise 4.4. $\int_0^1 t(1 + 0.7t) \, dt = \frac{1}{2} + \frac{0.7}{3}$, so $r(x) = 1 + 0.7x - 1 - x(\frac{1}{2} + \frac{0.7}{3}) = x(0.7 - 0.5 - 0.2\bar{3}) = -x/30$. Since $|\lambda| M_K = \frac{1}{2}$, Proposition 4.2 gives $\|e\|_\infty \leq \frac{1/30}{1/2} = \frac{1}{15}$. The actual error is $(1 + 0.75x) - (1 + 0.7x) = 0.05x$, with sup norm 0.05, inside the bound.

Exercise 4.5. The error is $-3\epsilon x/(3 - \lambda)$, so $\|e\|_\infty/\|r\|_\infty = 3/|3 - \lambda|$. This exceeds 100 exactly when $|3 - \lambda| < 0.03$. At $\lambda = 3$ the error equation has no unique solution at all, which is the extreme form of the same amplification.

Exercise 4.6. At $x = 0$, $u(0) = 0$. The Leibniz rule gives $u'(x) = 1 + \int_0^x u(t) dt$ (the boundary term vanishes), so $u'(0) = 1$, and differentiating again gives $u'' = u$. The solution is $\sinh x$. Substitution confirms it: $\int_0^x (x-t) \sinh t dt = \sinh x - x$, so the right side is $\sinh x$.

Exercise 4.7. Split the integral at $t = x$: $u(x) = \int_0^x t f(t) dt + x \int_x^1 f(t) dt$. Then $u(0) = 0$. By the Leibniz rule, $u'(x) = xf(x) + \int_x^1 f(t) dt - xf(x) = \int_x^1 f(t) dt$, so $u'(1) = 0$ and $u'' = -f$. For $f \equiv 1$, $u = x^2/2 + x(1-x) = x - x^2/2$, which has $u(0) = 0$, $u'(1) = 1 - 1 = 0$ and $-u'' = 1$.

Exercise 4.8. Differentiation discards the information in the values at $x = 0$. Setting $x = 0$ in the equation gives $u(0) = 0$, and setting $x = 0$ in $u'(x) = 1 + \int_0^x u(t) dt$ gives $u'(0) = 1$. The function e^x has $u(0) = 1$, so it fails. Its residual is

$$e^x - x - \int_0^x (x-t)e^t dt = e^x - x - (e^x - 1 - x) = 1,$$

using $\int_0^x (x-t)e^t dt = e^x - 1 - x$. The general solution of $u'' = u$ is $Ae^x + Be^{-x}$, and the conditions give $A + B = 0$, $A - B = 1$, so $u = \frac{1}{2}(e^x - e^{-x}) = \sinh x$, whose residual is 0 (Exercise 4.6).

Exercise 4.9. For $u = x^2/4$, $\sqrt{u(t)} = t/2$ for $t \geq 0$ and $\int_0^x t/2 dt = x^2/4$. For $u \equiv 0$ both sides vanish. For u_a : if $x \leq a$ the integrand is zero on $[0, x]$ and $u_a(x) = 0$; if $x > a$, $\int_0^x \sqrt{u_a} dt = \int_a^x (t-a)/2 dt = (x-a)^2/4$. Each u_a is continuous and verified by substitution, so verification cannot establish uniqueness here. The nonlinearity $\sqrt{u}$ is not Lipschitz at $u = 0$, which is the source of the failure (Chapter 16).

Exercise 4.10. The residual is $\sin(n\pi x)/(n\pi)^2$, with sup norm $1/(n\pi)^2$. The condition $1/(n\pi)^2 \leq 10^{-6}$ is $n\pi \geq 10^3$, that is $n \geq 318.31$, so the smallest integer is $n = 319$. For every such n the error is $-\sin(n\pi t)$, with sup norm 1.

Exercise 4.11. (a) Linear Fredholm, second kind, with $|\cos(xt)| \leq 1$ on $[0, 1]^2$, so $M_K \leq 1$ and $|\lambda|M_K \leq \frac{1}{10} < 1$: the Neumann series converges quickly with the bound of Proposition 4.2; a Nyström method is a good numerical alternative. (b) Volterra, first kind, weakly singular kernel of Abel type with $\alpha = \frac{1}{3}$. (c) Volterra, second kind, convolution kernel $k(y) = y^2$; the Laplace transform gives $U = 1/s + 2U/s^3$. Differentiating three times gives the equivalent problem $u''' = 2u$, $u(0) = 1$, $u'(0) = u''(0) = 0$. (d) Cauchy kernel, so the integral exists only as a principal value. (e) Fredholm, second kind, kernel $1 \cdot 1 + x \cdot t$ of rank 2: a 2×2 system for $\int u$ and $\int tu$.

Exercise 4.12. If u_1, u_2 are continuous solutions, $v = u_1 - u_2$ satisfies $v = \lambda T v$. By the row-sum bound, $\|v\|_\infty = |\lambda| \|T v\|_\infty \leq |\lambda| M_K \|v\|_\infty$, so $(1 - |\lambda| M_K) \|v\|_\infty \leq 0$. With $|\lambda| M_K < 1$ this gives $v = 0$.

For Benchmark B2, $M_K = \max_x \int_0^1 x t \, dt = \max_x x/2 = \frac{1}{2}$, so at $\lambda = 2$ the product is 1 and the proposition does not apply. Suppose $v = 2x \int_0^1 t v(t) \, dt$. Then $v = cx$ for the number $c = 2 \int_0^1 t v \, dt$, and substituting gives $c = 2c/3$, so $c = 0$ and $v = 0$. The homogeneous equation has only the zero solution, and the difference argument shows the inhomogeneous solution is unique. The smallness condition is sufficient, not necessary.

Exercise 4.13. The error equation (4.2) for this problem is $e(x) = \lambda \int_0^x e(t) \, dt - r(x)$. Put $\rho = \|r\|_\infty$ and $\phi(x) = \int_0^x |e(t)| \, dt$. Then

$$\phi'(x) = |e(x)| \leq \rho + |\lambda| \phi(x).$$

Multiplying by $e^{-|\lambda|x}$ gives $(e^{-|\lambda|x} \phi(x))' \leq \rho e^{-|\lambda|x}$. Integrating from 0 to x and using $\phi(0) = 0$,

$$e^{-|\lambda|x} \phi(x) \leq \frac{\rho}{|\lambda|} (1 - e^{-|\lambda|x}), \quad \text{so} \quad |\lambda| \phi(x) \leq \rho (e^{|\lambda|x} - 1)$$

for $\lambda \neq 0$ (for $\lambda = 0$, $e = -r$ directly). Therefore $|e(x)| \leq \rho + |\lambda| \phi(x) \leq \rho e^{|\lambda|x}$. The bound holds for every λ , with no smallness condition, because the Volterra structure lets the error grow only step by step in x (this is Gronwall's inequality). For Example 4.2, it gives $\|e\|_\infty \leq e/6 \approx 0.453$, which contains the actual 0.218.

Exercise 4.14. Since $\int_0^1 t \sin(40\pi t) \, dt = -1/(40\pi)$, the residual is

$$r(x) = u_c(x) - 1 - x \int_0^1 t u_c(t) \, dt = 0.01 \sin(40\pi x) + \frac{0.01 x}{40\pi}.$$

At $x = k/4$ the sine vanishes, so the 5-point residual is at most $0.01/(40\pi) \approx 7.96 \times 10^{-5}$, attained at $x = 1$. On the grid $x = k/1000$ the points come within 0.0005 of every peak of the sine, where $|\sin(40\pi x)|$ exceeds 0.998, and the largest residual is about 0.0100. The error is $e = -0.01 \sin(40\pi x)$, so $\|e\|_\infty = 0.01$. The 5-point check underestimates the error by a factor of more than 100, because those points happen to be zeros of the error. Residuals must be evaluated away from any special set of points.

Chapter 5 Separable Kernels and Finite-Dimensional Reduction

Exercise 5.1. Put $c = \int_0^1 t u(t) dt$, so $u(x) = e^x + \lambda cx$. Multiplying by x and integrating,

$$c = \int_0^1 t e^t dt + \lambda c \int_0^1 t^2 dt = 1 + \frac{\lambda c}{3},$$

since $\int_0^1 t e^t dt = [(t-1)e^t]_0^1 = 1$. For $\lambda \neq 3$, $c = 3/(3-\lambda)$ and $u(x) = e^x + 3\lambda x/(3-\lambda)$. Check: $\int_0^1 t u = 1 + \frac{3\lambda}{3-\lambda} \cdot \frac{1}{3} = \frac{3}{3-\lambda} = c$, so the right side equals $e^x + \lambda cx = u$. At $\lambda = 3$ the equation for c reads $0 \cdot c = 1$, which has no solution; equivalently, the kernel is symmetric with eigenfunction x and the compatibility integral $\int_0^1 x e^x dx = 1$ is not zero.

Exercise 5.2. With $a = b = \sin$, $A = \int_0^\pi \sin^2 t dt = \pi/2$ and $\beta = \int_0^\pi \sin t \cos t dt = 0$. The system is $(1 - \lambda\pi/2)c = 0$. For $\lambda \neq 2/\pi$, $c = 0$ and $u = \cos x$; indeed $\int_0^\pi \sin t \cos t dt = 0$ makes the integral term vanish. At $\lambda = 2/\pi$ every c works, giving $u = \cos x + \frac{2}{\pi} c \sin x$, that is $u = \cos x + C \sin x$ for arbitrary C . This agrees with Theorem 5.7: the kernel is symmetric, $\psi = \sin x$, and $\int_0^\pi \cos x \sin x dx = 0$, so the forcing term is compatible, and the family has one free constant because the eigenspace is one-dimensional.

Exercise 5.3. Take $a_1 = b_1 = 1$, $a_2 = x$, $b_2 = 3t$. Then

$$\begin{aligned} A_{11} &= \int_{-1}^1 1 dt = 2, & A_{12} &= \int_{-1}^1 t dt = 0, \\ A_{21} &= \int_{-1}^1 3t dt = 0, & A_{22} &= \int_{-1}^1 3t^2 dt = 2, \end{aligned}$$

so $A = 2I$ and $D(\lambda) = (1-2\lambda)^2$. The only characteristic value is $\lambda = \frac{1}{2}$, where $I - \lambda A = 0$ and every vector is a null vector; the eigenfunctions are all nonzero $C_1 + C_2 x$, a two-dimensional eigenspace. (The kernel is $P_0(x)P_0(t) + 3P_1(x)P_1(t)$ in Legendre polynomials, which explains the diagonal matrix.)

For the forced equation, $\beta_1 = \int_{-1}^1 t^2 dt = \frac{2}{3}$ and $\beta_2 = \int_{-1}^1 3t \cdot t^2 dt = 0$. For $\lambda \neq \frac{1}{2}$, $c_1 = \frac{2}{3}/(1-2\lambda)$ and $c_2 = 0$, so

$$u(x) = x^2 + \frac{2\lambda}{3(1-2\lambda)}.$$

Check: $\int_{-1}^1 (1 + 3xt)u(t) dt = \int_{-1}^1 u + 3x \int_{-1}^1 tu = \frac{2}{3} + \frac{4\lambda}{3(1-2\lambda)} + 0 = \frac{2}{3(1-2\lambda)}$, and $x^2 + \lambda \cdot \frac{2}{3(1-2\lambda)} = u(x)$. At $\lambda = \frac{1}{2}$ the kernel is symmetric, so the compatibility conditions are $\int_{-1}^1 x^2 \cdot 1 dx = 0$ and $\int_{-1}^1 x^2 \cdot x dx = 0$. The first fails, so there is no solution.

Exercise 5.4. With $a_1 = b_1 = \cos$, $a_2 = b_2 = \sin$ on $[0, \pi/2]$: $\int_0^{\pi/2} \cos^2 = \int_0^{\pi/2} \sin^2 = \pi/4$ and $\int_0^{\pi/2} \sin \cos = \frac{1}{2}$. So $D(\lambda) = (1 - \lambda\pi/4)^2 - \lambda^2/4$, which vanishes when $1 - \lambda\pi/4 = \pm\lambda/2$. The plus sign gives $\lambda = 4/(\pi + 2) \approx 0.778$; then the first row of $(I - \lambda A)c = 0$ reads $\frac{\lambda}{2}c_1 - \frac{\lambda}{2}c_2 = 0$, so $c_1 = c_2$ and the eigenfunction is $\cos x + \sin x$. The minus sign gives $\lambda = 4/(\pi - 2) \approx 3.504$ with $c_1 = -c_2$ and eigenfunction $\cos x - \sin x$. Check the first: $\int_0^{\pi/2} \cos(x-t)(\cos t + \sin t) dt = \cos x(\frac{\pi}{4} + \frac{1}{2}) + \sin x(\frac{1}{2} + \frac{\pi}{4})$, which is $\frac{\pi+2}{4}(\cos x + \sin x)$. On the full period of Example 5.5 the off-diagonal entries vanish and the 2 characteristic values merge into the double value $1/\pi$; on a quarter period they separate.

Exercise 5.5. (a) $e^{x+t} = e^x e^t$ has rank 1. (b) $(x-t)^2 = x^2 \cdot 1 + x \cdot (-2t) + 1 \cdot t^2$. The functions $1, x, x^2$ are independent and so are $1, -2t, t^2$; a representation with fewer terms would put every function $x \mapsto (x-t)^2$ in a space of dimension at most 2, but the three functions $x^2, (x-1)^2$ and $(x-\frac{1}{2})^2$ obtained at $t = 0, 1, \frac{1}{2}$ are independent. So the rank is 3. (c) $\cos(x+t) = \cos x \cos t - \sin x \sin t$; the functions $x \mapsto \cos(x+t)$ at $t = 0$ and $t = \pi/2$ are $\cos x$ and $-\sin x$, independent, so the rank is 2. (d) Rank 3 by the same argument with $1, x, x^2$.

(e) If $|x-t| = \sum_{i=1}^n a_i(x)b_i(t)$, every function $x \mapsto |x-t|$ would lie in $\text{span}\{a_1, \dots, a_n\}$. Pick $n+1$ distinct points $0 < t_1 < \dots < t_{n+1} < 1$. The functions $g_k(x) = |x-t_k|$ are independent: in a vanishing combination $\sum \gamma_k g_k = 0$, every g_j with $j \neq k$ is differentiable at t_k while g_k has a corner there, so $\gamma_k = 0$. That is $n+1$ independent functions in an n -dimensional space, a contradiction. (f) The same argument works with $g_k(x) = \sin(t_k x)$ for distinct $t_k \in (0, 1]$: their Maclaurin coefficients of x^{2m+1} are $(-1)^m t_k^{2m+1}/(2m+1)!$, and a vanishing combination would force $\sum_k \gamma_k t_k (t_k^2)^m = 0$ for $m = 0, \dots, n$, a Vandermonde system in the distinct numbers t_k^2 with only the zero solution.

Exercise 5.6. Here $a(x) = x$ and $b(t) = 2 - 3t$, so $A = \int_0^1 (2 - 3t)t dt = 1 - 1 = 0$ and $D(\lambda) = 1 - \lambda \cdot 0 = 1$, which never vanishes; by Proposition 5.6 there are no characteristic values. The coefficient equation is $c = \beta + \lambda \cdot 0 \cdot c$ with $\beta = \int_0^1 (2 - 3t) dt = \frac{1}{2}$, so $c = \frac{1}{2}$ and $u(x) = 1 + \lambda x/2$. Check: $\int_0^1 (2 - 3t)(1 + \lambda t/2) dt = \frac{1}{2} + \frac{\lambda}{2} \int_0^1 (2t - 3t^2) dt = \frac{1}{2} + \frac{\lambda}{2} \cdot 0 = \frac{1}{2}$, so the right side is $1 + \lambda x \cdot \frac{1}{2} = u(x)$.

Exercise 5.7. Write $\sin(x - t) = \sin x \cos t + (-\cos x) \sin t$, so $a_1 = \sin$, $b_1 = \cos$, $a_2 = -\cos$, $b_2 = \sin$. Over $[0, 2\pi]$,

$$A = \begin{pmatrix} \int \cos \sin & -\int \cos^2 \\ \int \sin^2 & -\int \sin \cos \end{pmatrix} = \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}, \quad D(\lambda) = \det \begin{pmatrix} 1 & \lambda\pi \\ -\lambda\pi & 1 \end{pmatrix} = 1 + \pi^2 \lambda^2.$$

This has no real roots; its roots are $\pm i/\pi$. With $f = x$: $\beta_1 = \int_0^{2\pi} t \cos t \, dt = 0$ and $\beta_2 = \int_0^{2\pi} t \sin t \, dt = -2\pi$. The system $c_1 + \lambda\pi c_2 = 0$, $-\lambda\pi c_1 + c_2 = -2\pi$ gives

$$c_2 = -\frac{2\pi}{1 + \pi^2 \lambda^2}, \quad c_1 = \frac{2\pi^2 \lambda}{1 + \pi^2 \lambda^2},$$

and $u = x + \lambda(c_1 \sin x - c_2 \cos x) = x + \lambda(2\pi^2 \lambda \sin x + 2\pi \cos x)/(1 + \pi^2 \lambda^2)$. Substituting back, $\int_0^{2\pi} \sin(x-t) t \, dt = 2\pi \cos x$, $\int_0^{2\pi} \sin(x-t) \sin t \, dt = -\pi \cos x$ and $\int_0^{2\pi} \sin(x-t) \cos t \, dt = \pi \sin x$; collecting terms reproduces $u - x$ divided by λ , so the residual is zero.

Exercise 5.8. For $K(x, t) = x^2 t$ we have $a(x) = x^2$, $b(t) = t$ and $A = \int_0^1 t \cdot t^2 \, dt = \frac{1}{4}$, so $\lambda = 4$ is indeed the characteristic value and $\varphi(x) = x^2$ is an eigenfunction. The error is in the compatibility test. Theorem 5.7 requires $\int_0^1 f \psi = 0$ where ψ solves the adjoint homogeneous equation with kernel $K(t, x) = t^2 x$, that is $\psi(x) = 4x \int_0^1 t^2 \psi(t) \, dt$, whose solutions are multiples of $\psi(x) = x$. The correct test gives

$$\int_0^1 x \left(x - \frac{3}{4}\right) dx = \frac{1}{3} - \frac{3}{8} = -\frac{1}{24} \neq 0,$$

so the equation has no solution. Direct confirmation: with $c = \int_0^1 t u(t) \, dt$, the equation says $u = x - \frac{3}{4} + 4cx^2$, and then $c = \int_0^1 t(t - \frac{3}{4}) \, dt + 4c \int_0^1 t^3 \, dt = -\frac{1}{24} + c$, which is the impossible statement $0 = -\frac{1}{24}$. The sheet tested f against φ , which is only valid for symmetric kernels.

Exercise 5.9. With $A_{ij} = \int a_i b_j$ the matrix is $\begin{pmatrix} 1/2 & 1/3 \\ 1 & 1/2 \end{pmatrix} = A^T$. At $\lambda = -1$ the system $(I + A^T)c = \beta$ with $\beta = (1, \frac{1}{2})$ reads $\frac{3}{2}c_1 + \frac{1}{3}c_2 = 1$ and $c_1 + \frac{3}{2}c_2 = \frac{1}{2}$, whose solution is $c_1 = \frac{16}{23}$, $c_2 = -\frac{3}{23}$; then $u = 1 - (c_1 x + c_2) = (26 - 16x)/23$, the reported answer. Now check it in the original equation: $\int_0^1 u = \frac{18}{23}$ and $\int_0^1 tu = \frac{1}{3}$, so $\int_0^1 (x+t)u(t) \, dt = \frac{18}{23}x + \frac{1}{3}$ and the right side of (5.10) is $1 - \frac{18}{23}x - \frac{1}{3} = \frac{2}{3} - \frac{18}{23}x$. That is not $u(x) = \frac{26}{23} - \frac{16}{23}x$, so the reported function is not a solution. The correct matrix $A_{ij} = \int b_i a_j$ gives $u = (18 - 12x)/23$ (Example 5.2). The characteristic values agree because a matrix and its transpose have the same determinant, so $\det(I - \lambda A^T) = \det(I - \lambda A)$; only the forced solution depends on the orientation of A .

Exercise 5.10. Suppose the b_i were dependent. Relabel so that $b_n = \sum_{i=1}^{n-1} \gamma_i b_i$ for some constants γ_i (some b_j is a combination of the others). Then

$$K(x, t) = \sum_{i=1}^{n-1} a_i(x)b_i(t) + a_n(x) \sum_{i=1}^{n-1} \gamma_i b_i(t) = \sum_{i=1}^{n-1} (a_i(x) + \gamma_i a_n(x))b_i(t),$$

a representation with $n - 1$ terms whose functions are still continuous. That contradicts the assumption that n is the smallest possible number of terms. (If $n = 1$, dependence means $b_1 = 0$, so $K = 0$, which has rank 0, again a contradiction.)

Exercise 5.11. Here $b_i = a_i$, so $A_{ij} = \int_a^b a_i(t)a_j(t) dt = A_{ji}$: A is symmetric. For a real vector c ,

$$c^T A c = \sum_{i,j} c_i c_j \int_a^b a_i a_j = \int_a^b \left(\sum_i c_i a_i(t) \right)^2 dt \geq 0,$$

with equality only if the continuous function $\sum c_i a_i$ vanishes identically, which by independence forces $c = 0$. So A is positive definite. A real symmetric positive definite matrix has n real positive eigenvalues $\mu_1, \dots, \mu_n$ (with multiplicity), and $D(\lambda) = \prod_k (1 - \lambda \mu_k)$. By Proposition 5.6 the characteristic values are $1/\mu_k > 0$.

Exercise 5.12. For rank one the coefficient system is the scalar equation $(1 - \lambda \kappa)c = \beta$ with $\beta = \int_a^b b(t)f(t) dt$. Hence $c = \beta/(1 - \lambda \kappa)$ and

$$u(x) = f(x) + \lambda c a(x) = f(x) + \lambda \int_a^b \frac{a(x)b(t)}{1 - \lambda \kappa} f(t) dt.$$

The function $R(x, t; \lambda) = a(x)b(t)/(1 - \lambda \kappa)$ is the resolvent kernel of Chapter 6; it has a pole exactly at the characteristic value $\lambda = 1/\kappa$. For Benchmark B2, $a(x) = x$, $b(t) = t$, $\kappa = \frac{1}{3}$, and $R = 3xt/(3 - \lambda)$.

Exercise 5.13. With $a_1 = 1$, $b_1 = 1$, $a_2 = x$, $b_2 = t$: $A = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}$ and $\beta = (1, \frac{1}{2})$. The system $(I - \frac{1}{2}A)c = \beta$ is

$$\frac{1}{2}c_1 - \frac{1}{4}c_2 = 1, \quad -\frac{1}{4}c_1 + \frac{5}{6}c_2 = \frac{1}{2},$$

with determinant $\frac{5}{12} - \frac{1}{16} = \frac{17}{48}$. Cramer's rule gives $c_1 = \frac{46}{17}$ and $c_2 = \frac{24}{17}$, so $u_2 = 1 + \frac{1}{2}(c_1 + c_2 x) = (40 + 12x)/17$. Check: $\int_0^1 u_2 = \frac{46}{17}$ and $\int_0^1 t u_2 = \frac{24}{17}$, so $1 + \frac{1}{2}(\frac{46}{17} + \frac{24}{17}x) = u_2$. Then $u_2(0) = \frac{40}{17} \approx 2.353$ and $u_2(1) = \frac{52}{17} \approx 3.059$. The errors are about 0.164 at $x = 0$ and 0.631 at $x = 1$; the maximum over $[0, 1]$ is at $x = 1$ and matches the $n = 2$ entry 6.31×10^{-1} of Table 5.1. Keeping only 2 Taylor terms underestimates the kernel, and the underestimate is largest at $x = t = 1$, which is where the solution error is largest.

Exercise 5.14. Suppose $T\varphi = \mu\varphi$ with $\varphi \neq 0$ and $\mu \neq 0$. Put $c_i = \int_a^b b_i \varphi$. Then $\mu\varphi = \sum_i c_i a_i$, so $c \neq 0$ (otherwise $\varphi = 0$). Multiplying $\mu\varphi(x) = \sum_j c_j a_j(x)$ by $b_i(x)$ and integrating gives $\mu c_i = \sum_j A_{ij} c_j$, so $Ac = \mu c$.

Conversely, suppose $Ac = \mu c$ with $c \neq 0$ and $\mu \neq 0$. Put $\varphi = \sum_j c_j a_j$. Then

$$(T\varphi)(x) = \sum_i a_i(x) \int_a^b b_i \varphi = \sum_i a_i(x) \sum_j A_{ij} c_j = \mu \sum_i c_i a_i(x) = \mu\varphi(x).$$

If φ were 0, then $(Ac)_i = \int b_i \varphi = 0$ for all i , so $\mu c = 0$ and $c = 0$, a contradiction. So φ is an eigenfunction.

The nonzero eigenvalues of T are defined without reference to any representation, and they coincide with the nonzero eigenvalues of A for every representation. The characteristic values are their reciprocals, so they are representation-independent. (The matrix A itself does depend on the representation, and a wasteful one can add eigenvalues equal to 0, which contribute no characteristic values.)

Exercise 5.15. Over $\mathbb{C}$, $\det(\lambda^{-1}I - A) = \prod_k (\lambda^{-1} - \mu_k)$, so $D(\lambda) = \det(I - \lambda A) = \prod_{k=1}^n (1 - \lambda\mu_k)$, where $\mu_1, \dots, \mu_n$ are the eigenvalues of A with multiplicity. D is a nonzero constant exactly when every $\mu_k = 0$, and otherwise it has the root $1/\mu_k$. By Exercise 5.14 the characteristic values are the reciprocals of the nonzero eigenvalues of A , so there are none exactly when all eigenvalues of A are 0, which for a square matrix is the same as A being nilpotent (Cayley–Hamilton gives $A^n = 0$).

For any g , $Tg = \sum_i a_i \gamma_i$ with $\gamma_i = \int b_i g$. Applying T again, $T^2 g = \sum_i a_i (A\gamma)_i$, and by induction $T^k g = \sum_i a_i (A^{k-1} \gamma)_i$ for $k \geq 1$. Since $A^n = 0$, $T^{n+1} = 0$. Now the finite sum $u = \sum_{k=0}^n \lambda^k T^k f$ satisfies $f + \lambda T u = f + \sum_{k=0}^n \lambda^{k+1} T^{k+1} f = f + \sum_{k=1}^n \lambda^k T^k f = u$, because the $k = n+1$ term vanishes. So this finite sum is the solution for every λ , and it is unique by Corollary 5.4 since $D(\lambda) \neq 0$. For Exercise 5.6, $A = 0$, $T^2 = 0$ and $u = f + \lambda T f = 1 + \lambda x/2$.

Chapter 6 Successive Approximations and the Neumann Series

Exercise 6.1. $K_2(x, t) = \int_0^1 (xs)(st) ds = xt \int_0^1 s^2 ds = xt/3$. If $K_n = xt/3^{n-1}$, then $K_{n+1} = \int_0^1 xs \cdot st/3^{n-1} ds = xt/3^n$, so $K_3 = xt/9$ and $K_n = xt/3^{n-1}$ by induction. Then

$$R = \sum_{n \geq 1} \lambda^{n-1} \frac{xt}{3^{n-1}} = xt \sum_{m \geq 0} \left(\frac{\lambda}{3}\right)^m = \frac{3xt}{3-\lambda},$$

a geometric series that converges exactly for $|\lambda| < 3$. The closed form agrees with the resolvent kernel of Chapter 5, which exists for all $\lambda \neq 3$.

Exercise 6.2. $K_2(x, t) = \int_0^1 (x+s)(s+t) ds = \int_0^1 (xt + (x+t)s + s^2) ds = xt + \frac{1}{2}(x+t) + \frac{1}{3}$. Unlike the kernels xt and e^{x-t} , the iterated kernel is not a multiple of K ; this happens because $x+t$ has rank 2 with 2 different operator eigenvalues, and the powers T^n mix them in different proportions.

Exercise 6.3. With $R = e^{x-t}/(1-\lambda)$, $u = x + \frac{\lambda}{1-\lambda} e^x \int_0^1 te^{-t} dt$, and $\int_0^1 te^{-t} dt = [-(t+1)e^{-t}]_0^1 = 1-2e^{-1}$. So $u = x + ce^x$ with $c = \lambda(1-2e^{-1})/(1-\lambda)$. Check: $\int_0^1 e^{x-t}(t+ce^t) dt = e^x(1-2e^{-1}) + ce^x$, and $x + \lambda e^x[(1-2e^{-1}) + c] = x + \lambda e^x(1-2e^{-1}) \cdot \frac{1}{1-\lambda} = x + ce^x = u$. The series derivation needs $|\lambda| < 1$; the checked formula holds for all $\lambda \neq 1$.

Exercise 6.4. $M = \max|K| = 1$ and $b-a = 1$, giving 1. $\int_0^1 |K(x, t)| dt = \sin(\pi x) \cdot \frac{2}{\pi}$, maximal at $x = \frac{1}{2}$, so $N_\infty = 2/\pi$ and the radius is $\pi/2$. $\|K\|_{\text{HS}}^2 = \int_0^1 \sin^2(\pi x) dx \int_0^1 \sin^2(\pi t) dt = \frac{1}{4}$, so $\|K\|_{\text{HS}} = \frac{1}{2}$ and the radius is 2. The kernel has rank one with $\kappa = \int_0^1 \sin^2(\pi t) dt = \frac{1}{2}$, so $K_n = \kappa^{n-1}K$ and the only characteristic value is 2; the actual radius is 2. For a symmetric rank-one kernel the Hilbert-Schmidt norm equals the operator eigenvalue in modulus, so that test is sharp here, while the row-norm test is not.

Exercise 6.5. $N_\infty(K) \leq M(b-a) \leq 1$, so $q \leq \frac{1}{2}$ and Proposition 6.8(a) gives $\|u - u_k\|_\infty \leq q^{k+1}/(1-q) \leq (\frac{1}{2})^{k+1}/\frac{1}{2} = 2^{-k}$. Since $2^{19} = 524288 < 10^6 < 2^{20} = 1048576$, the smallest k with $2^{-k} \leq 10^{-6}$ is $k = 20$. The actual number may be much smaller, as Example 6.3 shows.

Exercise 6.6. From $u = f + \lambda Tu$ and $u_k = f + \lambda Tu_{k-1}$ we get $e_k = \lambda Te_{k-1}$, so $\|e_k\|_\infty \leq q\|e_{k-1}\|_\infty$ by Lemma 6.5. Write $e_{k-1} = e_k + (u_k - u_{k-1})$ and use the triangle inequality: $\|e_k\| \leq q\|e_k\| + q\|u_k - u_{k-1}\|$, so $(1-q)\|e_k\| \leq q\|u_k - u_{k-1}\|$, which is (b). For the comparison, $u_k - u_{k-1} = \lambda^k T^k f$ by Lemma 6.1, and $\|\lambda^k T^k f\|_\infty \leq q^k \|f\|_\infty$. Hence $\frac{q}{1-q}\|u_k - u_{k-1}\|_\infty \leq \frac{q^{k+1}}{1-q}\|f\|_\infty$, which is the a priori bound (a). The a posteriori bound is therefore never worse, and it is much better whenever the actual differences shrink faster than q^k .

Exercise 6.7. Subtract $u = f + \lambda Tu$ from $v = f + \lambda Tv + r$ to get $v - u = \lambda T(v - u) + r$. Taking maximum norms and using Lemma 6.5, $\|v - u\|_\infty \leq q\|v - u\|_\infty + \|r\|_\infty$, so $\|v - u\|_\infty \leq \|r\|_\infty / (1 - q)$. The bound uses only v , the data and the kernel, so it applies to a candidate from a formula, a series, a computer or a guess. It does need the residual's true maximum over $[a, b]$, and it needs $q < 1$; near a characteristic value the factor $1/(1 - q)$, or its sharper replacement, can be very large, and then a small residual certifies little.

Exercise 6.8. The first error is logical: $|\lambda|M(b - a) < 1$ is a sufficient condition for convergence, so its failure proves nothing. In fact, by Example 6.1 the iterates are $u_k = 1 + 1.2x \sum_{j < k} 0.8^j$, a geometric series with ratio $\lambda/3 = 0.8$, which converges. The second error is worse: even a diverging Neumann series would say nothing about existence, because the series is one method among several (Proposition 6.9 and Example 6.1 with $\lambda = 4$). The equation is Benchmark B2 with $\lambda = 2.4 \neq 3$, so its unique solution is

$$u(x) = 1 + \frac{3(2.4)x}{2(3 - 2.4)} = 1 + 6x.$$

Check: $\int_0^1 t(1 + 6t) dt = \frac{1}{2} + 2 = \frac{5}{2}$, and $1 + 2.4x \cdot \frac{5}{2} = 1 + 6x$.

Exercise 6.9. Substitute: $1 + \lambda \int_0^x e^{2\lambda t} dt = 1 + \frac{1}{2}(e^{2\lambda x} - 1) = \frac{1}{2}(1 + e^{2\lambda x})$, which differs from $e^{\lambda x}$ for every $x > 0$ (their difference is $\frac{1}{2}(e^{\lambda x} - 1)^2$). The error is that $\mathcal{T}$ is not linear: $\mathcal{T}(g + h) \neq \mathcal{T}g + \mathcal{T}h$ because of the square, so Lemma 6.1 fails and the iterates are not $\sum \lambda^n \mathcal{T}^n 1$. (The computation $\mathcal{T}^n 1 = x^n/n!$ is also wrong: $\mathcal{T}1 = x$ but $\mathcal{T}x = x^3/3$.) The equation is Benchmark B5 with $a = 1$. Differentiating gives $u' = \lambda u^2$, $u(0) = 1$, so $u = 1/(1 - \lambda x)$, valid for $0 \leq x < 1/\lambda$. Check: $1 + \lambda \int_0^x (1 - \lambda t)^{-2} dt = 1 + [(1 - \lambda t)^{-1}]_0^x = 1/(1 - \lambda x)$.

Exercise 6.10. By induction,

$$K_{n+1}(x, t) = \int_a^b a(x)b(s) \kappa^{n-1} a(s)b(t) ds = \kappa^{n-1} a(x)b(t) \int_a^b b(s)a(s) ds = \kappa^n K(x, t).$$

Hence $N_\infty(K_n) = |\kappa|^{n-1}N_\infty(K)$ and $N_\infty(K_n)^{1/n} \rightarrow |\kappa|$. For $n \geq 1$, $\lambda^n T^n f = \lambda^n \kappa^{n-1} T f$. If $|\lambda\kappa| < 1$ this is a convergent geometric series (also by Theorem 6.10). If $|\lambda\kappa| \geq 1$ and $Tf \neq 0$, the terms do not tend to 0, so the series diverges. The actual solution $u = f + \lambda T f / (1 - \lambda\kappa)$ exists for all $\lambda \neq 1/\kappa$ (Chapter 5).

Exercise 6.11. Here $(Tg)(x) = \int_0^x g(t) dt$, and $T(x^n/n!) = x^{n+1}/(n+1)!$. Starting from $T^0 1 = 1$, induction gives $T^n 1 = x^n/n!$, so by Lemma 6.1

$$u_k(x) = \sum_{n=0}^k \frac{(\lambda x)^n}{n!},$$

the Taylor polynomial of $e^{\lambda x}$. Taylor's theorem with the Lagrange remainder gives, for $z = \lambda x$, $e^z - \sum_{n \leq k} z^n/n! = e^\xi z^{k+1}/(k+1)!$ for some ξ between 0 and z , so the error is at most $|z|^{k+1}e^{|z|}/(k+1)!$, which tends to 0 for every fixed z . Thus the iterates converge to $e^{\lambda x}$ for every λ , uniformly on each bounded interval $[0, b]$. Check: $1 + \lambda \int_0^x e^{\lambda t} dt = 1 + (e^{\lambda x} - 1) = e^{\lambda x}$. No characteristic value limits the convergence, because the factorial beats every power $|\lambda|^n$; Chapter 7 shows this is typical of Volterra equations.

Exercise 6.12. The table values are computed from the exact iterates, so a grid computation reproduces them up to the quadrature error of the trapezoid rule, which for this kernel (a corner on the diagonal, but at a grid node) and smooth iterates is of order h^2 with $h = 1/(N-1)$; with $N = 401$ the midpoint values agree with the table to about 4 significant digits, and the converged grid value at $\lambda = 9$ differs from $1/\cos(1.5)$ by about 7×10^{-4} . At $\lambda = 9$: $u_{20}(\frac{1}{2}) \approx 12.054$ and $u_{80}(\frac{1}{2}) \approx 14.129$, approaching $1/\cos(1.5) = 14.1368$ with error ratio about $9/\pi^2 \approx 0.912$ per step. At $\lambda = 12$: $u_{20}(\frac{1}{2}) \approx 351$ and $u_{80}(\frac{1}{2}) \approx 4.4 \times 10^7$, growing by about $12/\pi^2 \approx 1.216$ per step, while the equation has the solution (6.7) with $u(\frac{1}{2}) = 1/\cos(\sqrt{3}) \approx -6.228$. The divergence is a property of the method: 12 lies beyond the first characteristic value π^2 , but it is not itself a characteristic value, so the equation is uniquely solvable.

Exercise 6.13. Lemma 6.5 gives $\|T\|_\infty \leq N_\infty(K)$. The function $x \mapsto \int_a^b |K(x, t)| dt$ is continuous, so it attains its maximum $N_\infty(K)$ at some x_0 . For $\varepsilon > 0$ the function g_ε of the hint is continuous with $|g_\varepsilon| \leq 1$. Then

$$(Tg_\varepsilon)(x_0) = \int_a^b \frac{|K(x_0, t)|^2}{|K(x_0, t)| + \varepsilon} dt \geq \int_a^b (|K(x_0, t)| - \varepsilon) dt = N_\infty(K) - \varepsilon(b-a),$$

using $s^2/(s + \varepsilon) \geq s - \varepsilon$ for $s \geq 0$. So $\|T\|_\infty \geq \|Tg_\varepsilon\|_\infty \geq N_\infty(K) - \varepsilon(b-a)$ for every $\varepsilon > 0$, and hence $\|T\|_\infty \geq N_\infty(K)$. The sign function $K(x_0, t)/|K(x_0, t)|$ would give equality at once but is not continuous, which is why the ε is needed.

Exercise 6.14. By Theorem 6.6(a) the series $R = \sum_{n \geq 1} \lambda^{n-1} K_n$ converges uniformly on $[a, b]^2$, so it may be multiplied by the bounded function $K(x, s)$ and integrated term by term in s :

$$\begin{aligned} \lambda \int_a^b K(x, s) R(s, t; \lambda) \, ds &= \sum_{n \geq 1} \lambda^n \int_a^b K(x, s) K_n(s, t) \, ds \\ &= \sum_{n \geq 1} \lambda^n K_{n+1}(x, t) = R(x, t; \lambda) - K(x, t). \end{aligned}$$

For the second equation use $K_{n+1}(x, t) = \int_a^b K_n(x, s) K(s, t) \, ds$, which is the case $m = n$, second factor K_1 , of Proposition 6.3, and repeat the computation with $R(x, s; \lambda) K(s, t)$. For $K = xt$ on $[0, 1]$ you can check both equations with $R = 3xt/(3 - \lambda)$: $xt + \lambda \int_0^1 xs \cdot \frac{3st}{3 - \lambda} \, ds = xt(1 + \frac{\lambda}{3 - \lambda}) = \frac{3xt}{3 - \lambda}$.

Chapter 7 Volterra Equations and Resolvent Kernels

Exercise 7.1. From (7.2),

$$K_2(x, t) = \int_t^x xs \cdot st \, ds = xt \frac{x^3 - t^3}{3},$$

$$K_3(x, t) = \int_t^x xs \cdot st \frac{s^3 - t^3}{3} \, ds = xt \frac{(x^3 - t^3)^2}{18},$$

using $\int_t^x s^2(s^3 - t^3) \, ds = (x^3 - t^3)^2/6$. This is Proposition 7.6 with $a(x) = x$, $b(t) = t$ and $\Phi(x, t) = (x^3 - t^3)/3$, so $K_n = xt \Phi^{n-1}/(n-1)!$ and $R = xt e^{\lambda \Phi}$. Then

$$u(x) = x + \lambda \int_0^x xt e^{\lambda(x^3 - t^3)/3} t \, dt = x + x e^{\lambda x^3/3} \int_0^x \lambda t^2 e^{-\lambda t^3/3} \, dt$$

$$= x + x e^{\lambda x^3/3} (1 - e^{-\lambda x^3/3}) = x e^{\lambda x^3/3}.$$

The last integral uses $\frac{d}{dt}(-e^{-\lambda t^3/3}) = \lambda t^2 e^{-\lambda t^3/3}$ for $\lambda \neq 0$; for $\lambda = 0$ the answer $u = x$ is immediate. Check in the original equation: $x + \lambda x \int_0^x t \cdot t e^{\lambda t^3/3} \, dt = x + x(e^{\lambda x^3/3} - 1) = x e^{\lambda x^3/3}$.

Exercise 7.2. At $\lambda = -1$ the resolvent is $R(x, t; -1) = \sin(x - t)$, so

$$u(x) = x - \int_0^x \sin(x - t) t \, dt = x - (x - \sin x) = \sin x,$$

using $\int_0^x t \sin(x - t) \, dt = x - \sin x$ (integrate by parts once). Differentiating the equation twice gives $u' = 1 - \int_0^x u(t) \, dt$ and $u'' = -u$, with $u(0) = 0$ and $u'(0) = 1$ from the equation and its first derivative at $x = 0$. The unique solution of that problem is $\sin x$.

Exercise 7.3. (a) With $\lambda = 2$, $R = e^{3(x-t)}$ and $u(x) = 1 + 2 \int_0^x e^{3(x-t)} \, dt = 1 + \frac{2}{3}(e^{3x} - 1) = \frac{1}{3}(1 + 2e^{3x})$.

(b) With $\lambda = -1$, $R = 1$ and $u(x) = x - \int_0^x t \, dt = x - x^2/2$. Check: with $s = x - t$,

$$\int_0^x e^{x-t} \left(t - \frac{t^2}{2}\right) dt = \int_0^x e^s \left((x-s) - \frac{(x-s)^2}{2}\right) ds = \frac{x^2}{2},$$

by the formula $\int_0^x e^s(x-s)^m ds = m!(e^x - \sum_{k=0}^m x^k/k!)$ with $m = 1$ and $m = 2$: the two pieces are $e^x - 1 - x$ and $e^x - 1 - x - x^2/2$, and their difference is $x^2/2$. So the right side is $x - x^2/2 = u(x)$.

Exercise 7.4. Claim: $K_{m+n}(x, t) = \int_t^x K_m(x, s)K_n(s, t) ds$ for all $m, n \geq 1$. Induct on m . For $m = 1$ it is the definition (7.2). If it holds for m , then

$$K_{m+1+n}(x, t) = \int_t^x K(x, r)K_{m+n}(r, t) dr = \int_t^x K(x, r) \int_t^r K_m(r, s)K_n(s, t) ds dr.$$

The region is $t \leq s \leq r \leq x$; swapping the order,

$$K_{m+1+n}(x, t) = \int_t^x \left[\int_s^x K(x, r)K_m(r, s) dr \right] K_n(s, t) ds = \int_t^x K_{m+1}(x, s)K_n(s, t) ds.$$

All integrands are continuous on compact sets, so the swap is justified. With $n = 1$ we get $K_{m+1}(x, t) = \int_t^x K_m(x, s)K(s, t) ds$. Multiply by λ^m , sum over $m \geq 1$ and use uniform convergence of (7.7):

$$R(x, t; \lambda) - K(x, t) = \sum_{m \geq 1} \lambda^m K_{m+1}(x, t) = \lambda \int_t^x R(x, s; \lambda)K(s, t) ds.$$

Exercise 7.5. Let $\psi(x) = \alpha + \int_a^x \beta(t)\varphi(t) dt$ and $B(x) = \int_a^x \beta(t) dt$. Then $\varphi \leq \psi$ by hypothesis, and ψ is continuously differentiable with $\psi' = \beta\varphi \leq \beta\psi$ because $\beta \geq 0$. Hence

$$(e^{-B(x)}\psi(x))' = e^{-B(x)}(\psi'(x) - \beta(x)\psi(x)) \leq 0,$$

so $e^{-B(x)}\psi(x) \leq e^{-B(a)}\psi(a) = \alpha$. Therefore $\varphi(x) \leq \psi(x) \leq \alpha e^{B(x)}$. With β constant this is Lemma 7.7.

Exercise 7.6. (a) The kernel $\sin(x - t)$ is continuous on the triangle, so Theorem 7.4 applies for every λ .

(b) The integral runs over the fixed interval $[0, 1]$, so this is a Fredholm equation. Since $\sin(x - t) = \sin x \cos t - \cos x \sin t$, it has rank two. Writing $u = f + \lambda(c_1 \sin x - c_2 \cos x)$ with $c_1 = \int_0^1 \cos t u(t) dt$ and $c_2 = \int_0^1 \sin t u(t) dt$ leads (Chapter 5) to a 2×2 system with determinant

$$D(\lambda) = 1 + \lambda^2 \left(\frac{7 + \cos 4}{32} - \frac{\sin^4 1}{4} \right),$$

whose roots are $\lambda \approx \pm 3.7016 i$. At those values uniqueness fails and, for most f , existence fails too. For every real λ the solution exists and is unique, but that is a fact about this kernel, not a guarantee.

(c) Continuity is all Theorem 7.4 needs. The size of e^{10xt} only affects the constant in the bounds, which is $e^{|\lambda|e^{1000 \cdot 10}}$, a number of no practical use but finite.

(d) The kernel is not continuous on the triangle and not even integrable near the diagonal. For $u = 1$ the integral $\int_0^x (x-t)^{-1} dt$ diverges, so the equation does not define a map on $C[0, 1]$ and the theory of this chapter says nothing.

Exercise 7.7. There are two errors. First, a formula that is undefined at a parameter value tells you the derivation divided by zero there, not that the equation fails. At $\lambda = -1$ the resolvent is $R = e^{0 \cdot (x-t)} = 1$, and $u(x) = 1 - \int_0^x 1 dt = 1 - x$, which satisfies the equation (Example 7.4). The formula also has the limit $1 - x$ as $\lambda \rightarrow -1$, since $(e^{(1+\lambda)x} - 1)/(1 + \lambda) \rightarrow x$.

Second, the kernel e^{x-t} is continuous on the triangle, so Theorem 7.9 rules out characteristic values altogether: the homogeneous equation $u = \lambda Tu$ has only the zero solution for every λ . Even when a denominator does vanish at a genuine characteristic value of a Fredholm equation, the correct conclusion depends on compatibility (Chapter 12); it is never automatic that no solution exists.

Exercise 7.8. Here $K(x, t) = \cos(x - t)$, $K(x, x) = 1$, $\partial_x K = -\sin(x - t)$ and $f(0) = 0$, so Theorem 7.10 applies and gives

$$u(x) - \int_0^x \sin(x - t) u(t) dt = \cos x.$$

Try $u = 1$: $\int_0^x \sin(x - t) dt = 1 - \cos x$, and $1 - (1 - \cos x) = \cos x$. By uniqueness $u = 1$. Check in the original equation: $\int_0^x \cos(x - t) dt = \sin x$.

Exercise 7.9. If u is continuous, $g(x) = \int_0^x (x - t)u(t) dt$ has $g'(x) = \int_0^x u(t) dt$ by the Leibniz rule, so $g(0) = 0$, $g'(0) = 0$ and $g'' = u$. For $f(x) = x$ we would need $f'(0) = 0$, but $f'(0) = 1$. So no continuous solution exists.

In general a continuous solution forces $f(0) = 0$, $f'(0) = 0$ and $u = f''$. Conversely, if $f \in C^2$ with $f(0) = f'(0) = 0$, put $u = f''$. Then $g = \int_0^x (x - t)f''(t) dt$ satisfies $g'' = f''$ and $g(0) = g'(0) = 0$, so $g - f$ has zero second derivative and vanishes with its derivative at 0, which forces $g = f$. The solution is unique because the difference of two solutions has $u = 0'' = 0$.

Exercise 7.10. (a) The tail $\sum_{k>n} 10^k/k!$ first drops below 10^{-8} at $n = 40$ (Table 7.1, row $q = 10$). The actual error of the alternating partial sums is a little smaller and first drops below 10^{-8} at $n = 39$. The bound is honest and nearly sharp here.

(b) The terms $(-30)^k/k!$ grow until $k = 30$, where they reach about 7.8×10^{11} , and then decay. Double precision carries about 16 significant digits, so each large term carries a rounding error of order $10^{11} \cdot 10^{-16} = 10^{-5}$. The exact sum $e^{-30} \approx 9.4 \times 10^{-14}$ is far below

that noise level, and the computed value comes out near -6×10^{-5} , even with the wrong sign. The series converges mathematically for every λ , but summing it in floating point is useless when λ is large and negative. A marching method, as in the Computation box of Section 7.6, avoids the cancellation.

Exercise 7.11. The kernel $\sin(xt)$ is continuous on the triangle with $|K| \leq M = 1$, and $\|f\|_\infty = 1$. Theorem 7.4(d) gives $|u(x)| \leq e^{x/2}$ on $[0, 2]$, so $\|u\|_\infty \leq e^1 \approx 2.718$. Corollary 7.8 with $\|f - g\|_\infty = 10^{-3}$ gives $|u(x) - v(x)| \leq 10^{-3}e^{x/2} \leq 10^{-3}e \approx 2.7 \times 10^{-3}$. Both bounds are guarantees; the true values are smaller because $|\sin(xt)|$ is well below 1 for small xt .

Exercise 7.12. (a) By (7.4), $\|T^n\|^{1/n} \leq M(b-a)/(n!)^{1/n}$. From $e^n = \sum_k n^k/k! \geq n^n/n!$ we get $n! \geq (n/e)^n$, so $(n!)^{1/n} \geq n/e$ and $\|T^n\|^{1/n} \leq eM(b-a)/n \rightarrow 0$.

(b) For $K = 1$, $K_n(x, t) = (x-t)^{n-1}/(n-1)!$, so $|(T^n u)(x)| \leq \|u\|_\infty x^n/n! \leq \|u\|_\infty/n!$ on $[0, 1]$. For $u = 1$, $(T^n 1)(1) = 1/n!$, so the norm equals $1/n! > 0$.

(c) The series $\sum \lambda^n T^n f$ has terms bounded by $|\lambda|^n \|T^n\| \|f\|_\infty$, and $(|\lambda|^n \|T^n\|)^{1/n} = |\lambda| \|T^n\|^{1/n} \rightarrow 0 < 1$. By the root test the numerical series $\sum |\lambda|^n \|T^n\|$ converges, so the operator series converges absolutely in norm, for every λ .

Exercise 7.13. For each $b > a$, Theorem 7.4 gives a unique continuous solution u_b on $[a, b]$. If $b < b'$, the restriction of $u_{b'}$ to $[a, b]$ is a continuous solution on $[a, b]$, because the equation at a point $x \leq b$ only involves values on $[a, x]$. By uniqueness on $[a, b]$ it equals u_b . So $u(x) = u_b(x)$ for any $b \geq x$ is well defined, continuous and solves the equation on $[a, \infty)$. Any other continuous solution restricts to a solution on each $[a, b]$, hence equals u there.

For Benchmark B1 with $\lambda = 1$, $f = 1$ and $K = 1$ are bounded but $u = e^x$ is unbounded. Existence on every finite interval says nothing about boundedness on the half-line.

Chapter 8 Convolution and Transform Methods

Exercise 8.1. (a) $\int_0^x 1 \cdot t \, dt = x^2/2$; transform $\frac{1}{s} \cdot \frac{1}{s^2} = \frac{1}{s^3}$, which is $\mathcal{L}\{x^2/2\}$.

(b) $\int_0^x e^{a(x-t)} e^{bt} \, dt = e^{ax} \int_0^x e^{(b-a)t} \, dt = e^{ax} \frac{e^{(b-a)x} - 1}{b-a} = \frac{e^{ax} - e^{bx}}{a-b}$. The product of transforms is $\frac{1}{(s-a)(s-b)} = \frac{1}{a-b} \left(\frac{1}{s-a} - \frac{1}{s-b} \right)$, which inverts to the same function.

(c) With $\sin(x-t) \sin t = \frac{1}{2} [\cos(x-2t) - \cos x]$,

$$\int_0^x \sin(x-t) \sin t \, dt = \frac{1}{2} \left[\frac{\sin(x-2t)}{-2} \right]_0^x - \frac{x}{2} \cos x = \frac{1}{2} \sin x - \frac{x}{2} \cos x.$$

The transform is $1/(s^2+1)^2$, and the table entry for $\frac{1}{2}(\sin x - x \cos x)$ is the same (differentiate $\mathcal{L}\{\sin bx\} = b/(s^2+b^2)$ with respect to b at $b=1$ to get $\mathcal{L}\{x \cos x\} = (s^2-1)/(s^2+1)^2$).

Exercise 8.2. (a) $\int_{-\infty}^{\infty} 1 \cdot e^{-t^2} \, dt = \sqrt{\pi}$, independent of x . Here $u \in L^1$ and k is bounded, the first sufficient condition with the roles of k and u exchanged.

(b) $\int_{-\infty}^{\infty} 1 \, dt$ diverges. Both functions are continuous and bounded, which shows that continuity alone does not give a whole-line convolution.

(c) For fixed x and $|t| \geq |x|$, $e^{-|x-t|} e^{|t|/2} \leq e^{|x|} e^{-|t|} e^{|t|/2} = e^{|x|} e^{-|t|/2}$, which is integrable. The integral converges for every x even though u is unbounded.

(d) For $t \geq |x|$, $e^{-|x-t|} e^{2|t|} = e^{x-t} e^{2t} = e^{x+t}$, and $\int_{-\infty}^{\infty} e^{x+t} \, dt$ diverges.

Exercise 8.3. $F = 1/s$ and $K = 1/(s+1)$, so $U = \frac{1}{s} \cdot \frac{s+1}{s+1-\lambda}$. For $\lambda \neq 1$,

$$U = \frac{1}{1-\lambda} \cdot \frac{1}{s} + \frac{\lambda}{\lambda-1} \cdot \frac{1}{s-(\lambda-1)}, \quad u = \frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda} e^{(\lambda-1)x}.$$

For $\lambda = 1$, $U = (s+1)/s^2 = 1/s + 1/s^2$ and $u = 1+x$. As $\lambda \rightarrow 1$, write $\mu = \lambda - 1$: $u = (\lambda e^{\mu x} - 1)/\mu = (e^{\mu x} - 1)/\mu + e^{\mu x} \rightarrow x+1$. Check for $\lambda = 1$: $1 + \int_0^x e^{-(x-t)}(1+t) \, dt = 1+x$, since $\int_0^x e^{-s}(1+x-s) \, ds = x$. Nothing fails at $\lambda = 1$, as Chapter 7 guarantees.

Exercise 8.4. $F = 1/(s^2+1)$ and $K = s/(s^2+1)$, so

$$U = \frac{1}{s^2+1} \cdot \frac{1}{1-2s/(s^2+1)} = \frac{1}{s^2+1} \cdot \frac{s^2+1}{s^2-2s+1} = \frac{1}{(s-1)^2}.$$

The table entry $t^n e^{at}$ with $n = 1$, $a = 1$ gives $u(x) = xe^x$. Check: with integration by parts,

$$\int_0^x \cos(x-t) t e^t dt = \frac{1}{2}(xe^x - \sin x),$$

so the right side is $\sin x + 2 \cdot \frac{1}{2}(xe^x - \sin x) = xe^x$. The double root at $s = 1$ is what produces the factor x ; the forcing $\sin x$ has disappeared from the answer entirely, because its transform cancelled against the kernel's.

Exercise 8.5. Here $U = (s^2 + 1)F$, and Chapter 7 says a continuous solution needs $f(0) = f'(0) = 0$, since $K(x, x) = 0$ and $\partial_x K(x, x) = 1$; then $u = f'' + f$.

For $f = x^2$: $U = (s^2 + 1) \cdot 2/s^3 = 2/s + 2/s^3$, so $u = 2 + x^2 = f'' + f$.

For $f = x$: $U = (s^2 + 1)/s^2 = 1 + 1/s^2$. The constant 1 does not tend to zero, so there is no solution of exponential order; the formal answer is $\delta + x$. Directly, $f'(0) = 1 \neq 0$.

For $f = 1 - \cos x$: $U = (s^2 + 1)\left(\frac{1}{s} - \frac{s}{s^2 + 1}\right) = \frac{s^2 + 1}{s} - s = \frac{1}{s}$, so $u = 1$. Check: $\int_0^x \sin(x-t) dt = 1 - \cos x$.

Exercise 8.6. Transforming, $sU - 1 + 3U + 2U/s = 0$, so $U(s^2 + 3s + 2) = s$ and

$$U = \frac{s}{(s+1)(s+2)} = -\frac{1}{s+1} + \frac{2}{s+2}.$$

Hence $u = 2e^{-2x} - e^{-x}$. Check: $u(0) = 1$, $u' = -4e^{-2x} + e^{-x}$ and $\int_0^x u = (1 - e^{-2x}) - (1 - e^{-x}) = e^{-x} - e^{-2x}$, so $u' + 3u + 2 \int_0^x u = (-4 + 6 - 2)e^{-2x} + (1 - 3 + 2)e^{-x} = 0$.

Exercise 8.7. Transforming, $U = 1/s^2 + V/s$ and $V = 1/s - U/s$. Substituting the second into the first, $U = 1/s^2 + 1/s^2 - U/s^2$, so $U(1 + 1/s^2) = 2/s^2$ and $U = 2/(s^2 + 1)$. Then

$$V = \frac{1}{s} - \frac{2}{s(s^2 + 1)} = \frac{1}{s} - \frac{2}{s} + \frac{2s}{s^2 + 1} = -\frac{1}{s} + \frac{2s}{s^2 + 1}.$$

So $u = 2 \sin x$ and $v = 2 \cos x - 1$. Check: $x + \int_0^x (2 \cos t - 1) dt = x + 2 \sin x - x = 2 \sin x$, and $1 - \int_0^x 2 \sin t dt = 1 - 2(1 - \cos x) = 2 \cos x - 1$. Both kernels here are the constant 1, the causal convolution with 1 is integration, and the system is equivalent to $u' = 1 + v$, $v' = -u$ with $u(0) = 0$, $v(0) = 1$.

Exercise 8.8. Transforming $u = k + k * u$ gives $U = K + KU$, so $U = K/(1 - K)$.

(a) $K = 1/(s + 1)$ and $U = \frac{1/(s+1)}{s/(s+1)} = \frac{1}{s}$, so $u = 1$. Replacements happen at a constant rate from the start, which is the memoryless property of the exponential lifetime.

(b) $K = 1/(s+1)^2$ and

$$U = \frac{1}{(s+1)^2 - 1} = \frac{1}{s(s+2)} = \frac{1}{2} \left(\frac{1}{s} - \frac{1}{s+2} \right), \quad u = \frac{1}{2}(1 - e^{-2x}).$$

Check: $k + k * u = xe^{-x} + \frac{1}{2} \int_0^x (x-t)e^{-(x-t)}(1-e^{-2t}) dt$, and the integral equals $1 - e^{-2x} - 2xe^{-x}$, so the right side is $\frac{1}{2}(1 - e^{-2x})$. At $x = 0$ the density is 0: a new component almost never fails immediately under this lifetime law.

(c) $\int_0^\infty te^{-t} dt = 1$ and $\int_0^\infty t^2 e^{-t} dt = 2$. The limits 1 and $\frac{1}{2}$ are the reciprocals of the mean lifetimes, as the renewal theorem predicts: in the long run, replacements occur once per mean lifetime.

Exercise 8.9. Substituting $u = -\sin x - \cos x$: because $\frac{d}{dt}(e^{-t} \cos t) = -e^{-t}(\sin t + \cos t)$,

$$\int_0^x e^{x-t}(-\sin t - \cos t) dt = e^x [e^{-t} \cos t]_0^x = \cos x - e^x.$$

It misses $\cos x$ by e^x .

The error is in “which means $u = f' - f$ ”. The identity $\mathcal{L}\{f'\} = sF - f(0)$ gives $(s-1)F = \mathcal{L}\{f' - f\} + f(0)$, and here $f(0) = 1$. The extra constant 1 in U is the transform of a unit impulse at $t = 0$, not of any function, and by (8.3) no function of exponential order has a transform tending to 1. The correct conclusion is that the equation has no continuous solution, in agreement with the compatibility condition $f(0) = 0$ of the reduction theorem in Chapter 7.

Exercise 8.10. Let $|k|, |u| \leq Ce^{\gamma t}$ and $s > \gamma$. On the region $D = \{0 \leq t \leq x < \infty\}$,

$$|e^{-sx}k(x-t)u(t)| \leq C^2 e^{-sx} e^{\gamma(x-t)} e^{\gamma t} = C^2 e^{-(s-\gamma)x},$$

and $\iint_D C^2 e^{-(s-\gamma)x} dt dx = C^2 \int_0^\infty x e^{-(s-\gamma)x} dx = C^2/(s-\gamma)^2 < \infty$. So the function is absolutely integrable on D , and Fubini's theorem allows either order. Integrating in x first over $x \geq t$ and substituting $y = x - t$,

$$\int_0^\infty u(t) \int_t^\infty e^{-sx} k(x-t) dx dt = \int_0^\infty u(t) e^{-st} dt \int_0^\infty e^{-sy} k(y) dy = U(s)K(s).$$

Integrating in t first gives $\int_0^\infty e^{-sx}(k * u)(x) dx = \mathcal{L}\{k * u\}(s)$.

Exercise 8.11. (a) Write $(Ku)(x) = \int_0^x e^{-(x-t)}u(t) dt + \int_x^1 e^{-(t-x)}u(t) dt$. Differentiating, the boundary terms $u(x)$ and $-u(x)$ cancel and

$$(Ku)'(x) = - \int_0^x e^{-(x-t)}u(t) dt + \int_x^1 e^{-(t-x)}u(t) dt.$$

Differentiating again, the boundary terms are $-u(x)$ and $-u(x)$, so $(Ku)'' = Ku - 2u$. At $x = 0$ the first integral in $(Ku)'$ is empty, so $(Ku)'(0) = (Ku)(0)$; at $x = 1$ the second is empty, so $(Ku)'(1) = -(Ku)(1)$. Now $w = u - f = \lambda Ku$ gives $w'' = \lambda(Ku - 2u) = w - 2\lambda(f + w)$, which is $-w'' + (1 - 2\lambda)w = 2\lambda f$, and the boundary relations carry over to w .

(b) For $f = 1$, $\lambda = -\frac{1}{4}$: $-w'' + \frac{3}{2}w = -\frac{1}{2}$, with particular solution $-\frac{1}{3}$. The boundary conditions are symmetric under $x \mapsto 1 - x$, so $w = -\frac{1}{3} + A \cosh(\beta(x - \frac{1}{2}))$ with $\beta = \sqrt{3/2}$. The condition $w'(0) = w(0)$ reads $-A\beta \sinh(\beta/2) = -\frac{1}{3} + A \cosh(\beta/2)$, so $A = \frac{1}{3} / (\cosh(\beta/2) + \beta \sinh(\beta/2)) \approx 0.1674$. Then $u = 1 + w = \frac{2}{3} + A \cosh(\beta(x - \frac{1}{2}))$, with $u(0) = u(1) \approx 0.8665$ and $u(\frac{1}{2}) \approx 0.8341$. On the whole line the constant forcing gives the constant $1/(1 - 2\lambda) = \frac{2}{3}$. On $[0, 1]$ the negative feedback has less kernel mass to act through, most of all near the ends, so u stays above $\frac{2}{3}$ and is largest at the ends.

Exercise 8.12. (a) Substituting $t = x - \tau$, $\int_{-\infty}^{\infty} e^{-|x-t|} e^{i\omega t} dt = e^{i\omega x} \int e^{-|\tau|} e^{-i\omega \tau} d\tau = e^{i\omega x} \hat{k}(\omega) = \frac{2e^{i\omega x}}{1+\omega^2}$. Since $1 + \omega^2 = 2\lambda$, this is $e^{i\omega x}/\lambda$, so $e^{i\omega x} = \lambda k \star e^{i\omega x}$. Taking real and imaginary parts gives the claims for $\cos \omega x$ and $\sin \omega x$. These are bounded but not integrable.

(b) If $u \in L^1$ solves the equation, then (8.9) holds for every real ξ , and $\hat{u}$ is a bounded continuous function. At $\xi = \pm\omega$, $1 - \lambda \hat{k}(\pm\omega) = 1 - 2\lambda/(1+\omega^2) = 0$, so $\hat{f}(\pm\omega) = 0 \cdot \hat{u}(\pm\omega) = 0$.

(c) For $f = e^{-|x|}$, $\hat{f}(\omega) = 2/(1 + \omega^2) = 1/\lambda \neq 0$, so by (b) no L^1 solution exists. The situation mirrors the Fredholm alternative of Chapter 12: homogeneous solutions exist, and the data must be orthogonal, in the Fourier sense, to them.

Exercise 8.13. (a) $\hat{g}(\xi) = e^{-\xi^2/2}$, which lies in $(0, 1]$. So $1 - \lambda \hat{g}(\xi) \geq 1 - \lambda > 0$, and dividing (8.9) is legitimate; uniqueness in L^1 follows as in Example 8.7.

(b) Since $0 < \lambda \hat{g} \leq \lambda < 1$,

$$\frac{1}{1 - \lambda \hat{g}} = \sum_{n \geq 0} \lambda^n \hat{g}^n, \quad \hat{g}(\xi)^n = e^{-n\xi^2/2} = \hat{g}_n(\xi),$$

where $g_n(x) = e^{-x^2/(2n)}/\sqrt{2\pi n}$. The series converges uniformly in ξ , and each term corresponds to $\lambda^n g_n \star f$ with $\|g_n \star f\|_1 \leq \|f\|_1$, so $u = f + \sum_{n \geq 1} \lambda^n g_n \star f$ converges in L^1 .

(c) The transform did its job: it proved existence and uniqueness and gave a convergent representation. What it did not give is a closed form, because $1/(1 - \lambda e^{-\xi^2/2})$ is not the transform of anything elementary. For a Gaussian f each term is a Gaussian and the sum is a series of Gaussians, which is the Neumann series of Chapter 6 in disguise. For most f the practical route is numerical.

Exercise 8.14. At node $x_i = ih$ the trapezoid rule gives

$$u_i = 1 + h \left(\frac{1}{2} e^{-x_i^2} u_0 + \sum_{j=1}^{i-1} e^{-(x_i - x_j)^2} u_j \right) + \frac{h}{2} u_i,$$

which is solved for u_i one node at a time. The results at $x = 1$ are about 2.277946, 2.277881, 2.277866 and 2.277862 for the four step sizes. Successive differences are about 6.5×10^{-5} , 1.5×10^{-5} and 3.8×10^{-6} , with ratios close to 4, so the error behaves like Ch^2 , the order of the trapezoid rule for a smooth solution. A much finer grid gives $u(1) \approx 2.277861$. The transform told us the answer exists and is unique; the marching scheme told us what it is.

Chapter 9 Initial-Value Problems as Integral Equations

Exercise 9.1. Integrate from 0 to x and use $u(0) = 2$: $u(x) = 2 + \int_0^x 3t^2 u(t) dt$. The kernel $K(x, t) = 3t^2$ depends only on t , as it always does for a first-order linear problem, and the forcing term is the constant 2. For $u = 2e^{x^3}$,

$$2 + \int_0^x 3t^2 \cdot 2e^{t^3} dt = 2 + 2[e^{t^3}]_0^x = 2 + 2e^{x^3} - 2 = 2e^{x^3}.$$

Exercise 9.2. Here $p = -3$, $q = 2$, $g = 0$, $\alpha = 0$, $\beta = 1$ and $p' = 0$. Formula (9.8) gives $K_B(x, t) = -p + (x - t)(0 - q) = 3 - 2(x - t)$ and $f_B(x) = 0 + (1 + (-3) \cdot 0)x = x$. So

$$u(x) = x + \int_0^x [3 - 2(x - t)]u(t) dt.$$

For $u = e^{2x} - e^x$: $\int_0^x u = \frac{1}{2}e^{2x} - e^x + \frac{1}{2}$ and, by the Cauchy formula, $\int_0^x (x - t)u(t) dt = \frac{1}{4}e^{2x} - e^x + \frac{3}{4} + \frac{1}{2}x$. Then

$$x + 3\left(\frac{1}{2}e^{2x} - e^x + \frac{1}{2}\right) - 2\left(\frac{1}{4}e^{2x} - e^x + \frac{3}{4} + \frac{1}{2}x\right) = e^{2x} - e^x,$$

as required. The differential equation also holds, since $4e^{2x} - e^x - 3(2e^{2x} - e^x) + 2(e^{2x} - e^x) = 0$.

Exercise 9.3. With $p = -3$, $q = 2$, $g = 0$, $\alpha = 0$, $\beta = 1$, formula (9.7) gives $K_A(x, t) = 3 - 2(x - t)$ and $f_A(x) = 0 - 1 \cdot (-3) - 2(0 + x) = 3 - 2x$. The kernel equals K_B because the coefficients are constant. For $w = 4e^{2x} - e^x$: $\int_0^x w = 2e^{2x} - e^x - 1$ and $\int_0^x (x - t)w(t) dt = e^{2x} - e^x - x$. Then

$$3 - 2x + 3(2e^{2x} - e^x - 1) - 2(e^{2x} - e^x - x) = 4e^{2x} - e^x = w(x).$$

Exercise 9.4. By Theorem 9.6 with $b_0 = 0$, $b_1 = 1$ (or directly by the Leibniz rule): $u'(x) = 1 + \int_0^x u(t) dt$ and $u''(x) = u(x)$. Setting $x = 0$ in the equation and in the formula for u' gives $u(0) = 0$ and $u'(0) = 1$. The solution is $u = \sinh x$. Check: $\int_0^x (x - t) \sinh t dt = \sinh x - x$, and $x + \sinh x - x = \sinh x$.

Exercise 9.5. By the lemma with $n = 3$, the triple integral equals $\int_0^x \frac{(x-t)^2}{2} \cos t \, dt$. Directly: $\int_0^r \cos t \, dt = \sin r$, $\int_0^s \sin r \, dr = 1 - \cos s$, $\int_0^x (1 - \cos s) \, ds = x - \sin x$. Through the single integral, two integrations by parts give the same value $x - \sin x$; equivalently, (9.5) with $u = \sin$ and $n = 3$ gives $\sin x = x + \int_0^x \frac{1}{2}(x-t)^2(-\cos t) \, dt$.

Exercise 9.6. Here $p = 2$, $q = 0$, $\alpha = 1$, $\beta = -2$. In (9.8) the forcing term is $f_B(x) = \alpha + [\beta + p(a)\alpha](x-a) = 1 + (-2+2)x = 1$ and the kernel is $K_B = -p = -2$. The student used $\beta x = -2x$ and dropped $p(a)\alpha x = 2x$, the boundary term from integrating $\int_0^x p u'$ by parts. The correct equation $u = 1 - 2 \int_0^x u$ gives $u' = -2u$, $u(0) = 1$, so $u = e^{-2x}$, which does satisfy $u'' + 2u' = 0$ with $u'(0) = -2$.

The student's equation differentiates to $u' = -2 - 2u$ with $u(0) = 1$, whose solution is $u = -1 + 2e^{-2x}$. It has $u'(0) = -4$, not -2 . The mistake is invisible in the kernel and shows up only when the initial conditions are checked, which is why the check has to be done in the original problem.

Exercise 9.7. The equation is $u'''' - u = 0$, so $a_1 = a_2 = a_3 = 0$, $a_4 = -1$, $g = 0$. Then $K(x, t) = -a_4(x-t)^3/3! = (x-t)^3/6$ and $f = -a_4 P_4 = P_4 = c_0 + c_1 x + c_2 x^2/2 + c_3 x^3/6 = 1 - x^2/2$. So $w(x) = 1 - \frac{1}{2}x^2 + \int_0^x \frac{1}{6}(x-t)^3 w(t) \, dt$. The solution of the differential problem is $u = \cos x$, so $w = u'''' = \cos x$. The check is Taylor's formula (9.5) with $n = 4$ applied to $\cos$: $\int_0^x \frac{1}{6}(x-t)^3 \cos t \, dt = \cos x - 1 + \frac{1}{2}x^2$, and $1 - \frac{1}{2}x^2$ plus this equals $\cos x$.

Exercise 9.8. By (9.13) with $\mathbf{g} = 0$ the equation is $\mathbf{u}(x) = (1, 0)^T + \int_0^x \mathbf{A} \mathbf{u}(t) \, dt$. For the first component, $\int_0^x (u_1 + u_2) \, dt = \int_0^x (\frac{3}{2}e^{3t} - \frac{1}{2}e^{-t}) \, dt = \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} - 1$, and adding 1 gives u_1 . For the second, $\int_0^x (4u_1 + u_2) \, dt = \int_0^x (3e^{3t} + e^{-t}) \, dt = e^{3x} - e^{-x}$, which is u_2 .

Exercise 9.9. The kernel has the form in Theorem 9.6 with $n = 2$, $b_0 = 1$, $b_1 = 2$. The theorem gives $u'' = f'' + (b_0 u)' + b_1 u = u' + 2u$ with $u(0) = f(0) = 1$ and $u'(0) = f'(0) + b_0 u(0) = 0 + 1 = 1$. The characteristic roots of $r^2 - r - 2 = 0$ are 2 and -1 , so $u = Ae^{2x} + Be^{-x}$ with $A + B = 1$ and $2A - B = 1$, giving $A = \frac{2}{3}$, $B = \frac{1}{3}$. A direct substitution into the integral equation confirms the result.

Exercise 9.10. The test is whether the x -derivatives of K satisfy a linear relation (9.17) with coefficients depending on x only.

- (a) $\partial_x^2 \sin(x-t) = -\sin(x-t)$, so $n = 2$ works. Differentiating twice gives $u' = f' + \int \cos(x-t)u$ and $u'' = f'' + u - \int \sin(x-t)u = f'' + u - (u - f)$, that is, $u'' = f'' + f$, with $u(0) = f(0)$ and $u'(0) = f'(0)$.
- (b) $\partial_x^2(xt) = 0$, so $n = 2$ works: $u' = f' + x^2 u + \int_0^x t u$ and $u'' = f'' + 2xu + x^2 u' + xu$.

- (c) $\partial_x^k K = (-1)^k H_k(x-t)e^{-(x-t)^2}$, and for fixed x these are linearly independent functions of t because H_k has degree k . No relation of the form (9.17) exists and the route never terminates.
- (d) $\partial_x^k K = (-1)^k k! (1+x-t)^{-k-1}$. For fixed x the functions $(1+x-t)^{-k-1}$, $k = 0, 1, 2, \dots$, are linearly independent in t , so again no finite relation exists.

Cases (c) and (d) are genuine Volterra equations and are best solved as such, by the resolvent methods of Chapter 7 or the numerical methods of Chapter 19.

Exercise 9.11. Let $J(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} g(t) dt$. For $n \geq 2$ the integrand vanishes at $t = x$, so the Leibniz rule gives $J'(x) = \int_a^x \frac{(x-t)^{n-2}}{(n-2)!} g(t) dt$. Repeating, $J^{(k)}(x) = \int_a^x \frac{(x-t)^{n-1-k}}{(n-1-k)!} g(t) dt$ for $0 \leq k \leq n-1$, and in particular $J^{(n-1)} = \int_a^x g$, so $J^{(n)} = g$. Each $J^{(k)}$ with $k \leq n-1$ vanishes at $x = a$. The repeated integral $I^n g$ has the same properties: $(I^n g)^{(k)} = I^{n-k} g$ vanishes at a for $k \leq n-1$, and $(I^n g)^{(n)} = g$. The difference $D = J - I^n g$ therefore satisfies $D^{(n)} = 0$ with $D(a) = \dots = D^{(n-1)}(a) = 0$, so D is a polynomial of degree less than n with all derivatives zero at a , hence $D \equiv 0$.

Exercise 9.12. Let $w \in C[a, b]$ and define u by the formula. By the Leibniz rule (the integrand vanishes on the diagonal), $u'(x) = \beta + \int_a^x w(t) dt$, which is continuously differentiable with $u'' = w$. So $u \in C^2[a, b]$, $u(a) = \alpha$ and $u'(a) = \beta$. Now

$$\begin{aligned} u'' + pu' + qu - g &= w - \left[g - p\beta - q(\alpha + \beta(x-a)) - p \int_a^x w - q \int_a^x (x-t)w \right] \\ &= w - f_A - \int_a^x K_A(x, t)w(t) dt. \end{aligned}$$

The left side vanishes for all x if and only if the right side does, that is, if and only if w solves (9.7). Conversely, any C^2 solution u of (9.6) is recovered from $w = u''$ by Taylor's formula (9.5), so the correspondence is one-to-one.

Exercise 9.13. The table in the text lists $u_{20}(1) = 1.33133419$ and $u_{40}(1) = 1.33131539$ against $u(1) = 1.33130912$, errors 2.51×10^{-5} and 6.28×10^{-6} , ratio 3.99. The trapezoidal rule has local error proportional to h^2 times the second derivative of the integrand, and because the method is stable for this kernel the global error is also proportional to h^2 . With $f = 1$ the equation is $u = 1 - \int_0^x t u$, whose solution is $e^{-x^2/2}$ (Example 9.4); the computed values approach $e^{-1/2} = 0.6065307 \dots$ with errors again shrinking by a factor near 4 per halving of h .

Chapter 10 Boundary-Value Problems and Green's Functions

Exercise 10.1. The left condition $u'(0) = 0$ is satisfied by $y_1 = 1$ and the right condition $u(1) = 0$ by $y_2 = 1 - x$. Then $C = p(y_1 y_2' - y_1' y_2) = 1 \cdot (-1) - 0 = -1$ and $G = -y_1(x_<)y_2(x_>)/C = 1 - x_> = 1 - \max(x, t)$.

Checks: (i) $1 - \max(x, t)$ is continuous. (ii) For $x < t$ it equals $1 - t$, constant in x ; for $x > t$ it equals $1 - x$; both solve $y'' = 0$. (iii) For $x < t$, $\partial_x G = 0$, so $\partial_x G(0, t) = 0$; for $x > t$, $G(1, t) = 0$. (iv) The slope is 0 to the left of t and -1 to the right, a jump of -1 . Symmetry is evident.

For $f = 1$: $u(x) = \int_0^x (1 - x) dt + \int_x^1 (1 - t) dt = x(1 - x) + \frac{1}{2}(1 - x)^2 = \frac{1}{2}(1 - x^2)$, and indeed $-u'' = 1$, $u'(0) = 0$, $u(1) = 0$.

Exercise 10.2. Split at $t = x$:

$$u(x) = (1 - x) \int_0^x t^3 dt + x \int_x^1 (1 - t)t^2 dt = \frac{(1 - x)x^4}{4} + x\left(\frac{1}{12} - \frac{x^3}{3} + \frac{x^4}{4}\right) = \frac{x - x^4}{12}.$$

Check: $u'' = -x^2$, so $-u'' = x^2$, and $u(0) = u(1) = 0$.

Exercise 10.3. The line $\ell = 1 + 2x$ carries the boundary data and $-\ell'' = 0$. The remainder $v = u - \ell$ solves $-v'' = 1$ with zero boundary values, so $v = \int_0^1 G(x, t) dt = \frac{1}{2}x(1 - x)$ by Example 10.1. Hence $u = 1 + 2x + \frac{1}{2}x(1 - x)$; check $-u'' = 1$, $u(0) = 1$, $u(1) = 1 + 2 + 0 = 3$.

Exercise 10.4. Here $f = 0$, $p = 1$ and $G_k(x, 0) = G_k(x, 1) = 0$, so (10.9) reduces to $u(x) = u(0) \partial_t G_k(x, 0) - u(1) \partial_t G_k(x, 1) = -\partial_t G_k(x, 1)$. For $t > x$, $G_k = \sinh(kx) \sinh(k(1 - t))/(k \sinh k)$, so $\partial_t G_k(x, t) = -\sinh(kx) \cosh(k(1 - t))/\sinh k$, which at $t = 1$ equals $-\sinh(kx)/\sinh k$. Hence $u = \sinh(kx)/\sinh k$. Check: $u'' = k^2 u$, $u(0) = 0$, $u(1) = 1$.

Exercise 10.5. Take $y_1 = x$ for the left condition. For the right condition we need $y_2'(1) + y_2(1) = 0$ with y_2 linear; $y_2 = 2 - x$ works, since $-1 + 1 = 0$. Then $C = x(-1) - 1 \cdot (2 - x) = -2$ and $G = \frac{1}{2}x_<(2 - x_>)$. Check the jump: the slope is $\frac{1}{2}(2 - t)$ for $x < t$ and $-\frac{1}{2}t$ for $x > t$, a difference of -1 . For $f = 1$,

$$u(x) = \frac{2-x}{2} \int_0^x t \, dt + \frac{x}{2} \int_x^1 (2-t) \, dt = \frac{(2-x)x^2}{4} + \frac{x}{2} \left(\frac{3}{2} - 2x + \frac{x^2}{2} \right) = \frac{3}{4}x - \frac{1}{2}x^2.$$

Check: $-u'' = 1$, $u(0) = 0$, $u'(1) + u(1) = (\frac{3}{4} - 1) + (\frac{3}{4} - \frac{1}{2}) = 0$.

Exercise 10.6. The homogeneous equation $y'' = -\omega^2 y$ has $y_1 = \sin(\omega x)$ with $y_1(0) = 0$ and $y_2 = \sin(\omega(1-x))$ with $y_2(1) = 0$. Then

$$C = \sin(\omega x)(-\omega \cos(\omega(1-x))) - \omega \cos(\omega x) \sin(\omega(1-x)) = -\omega \sin \omega,$$

so $G = \sin(\omega x_<) \sin(\omega(1-x_>))/(\omega \sin \omega)$. The construction needs $C \neq 0$, that is, $\sin \omega \neq 0$. When $\omega = n\pi$, $y_2 = \sin(n\pi - n\pi x) = \pm \sin(n\pi x)$ is a multiple of y_1 , and $\sin(n\pi x)$ is a nonzero solution of the homogeneous problem; by Theorem 10.8, the forced problem then has a solution only when $\int_0^1 f(x) \sin(n\pi x) \, dx = 0$. These $\omega^2 = (n\pi)^2$ are the characteristic values of Benchmark B3.

Exercise 10.7. The function $xt - \min(x, t)$ is the negative of the correct Green's function. It is continuous; for fixed t it is linear on each side of $x = t$, so it solves $-y'' = 0$ there; and it vanishes at $x = 0$ and $x = 1$. Those are three of the four checks, and they are all insensitive to an overall sign. The jump check fails: the slope is $t - 1$ for $x < t$ and t for $x > t$, so the jump is $+1$, while (10.6) with $p = 1$ requires -1 .

The error came from the sign convention. The operator in (10.1) is $-u''$, so the jump must be $-1/p$; a jump of $+1$ belongs to the operator $+u''$. Used on $f = 1$, the wrong kernel gives $u = -\int_0^1 G \, dt = -\frac{1}{2}x(1-x)$, which has $u'' = 1$. The boundary values are still right, which is exactly why a check of the boundary conditions alone would not catch the error.

Exercise 10.8. Fix $a < s < t < b$ and put $u = G(\cdot, s)$, $v = G(\cdot, t)$. Apply (10.5) on $[a, s]$, $[s, t]$ and $[t, b]$, where both functions are C^2 up to the endpoints (one-sided) and satisfy $Ly = 0$. The integrals vanish, so the sum of the boundary terms $[p(uv' - u'v)]$ over the three pieces is zero. The terms at a and b vanish because u and v satisfy the same separated boundary conditions (Lemma 10.2). At $x = s$, v is smooth and u has the slope jump $-1/p(s)$; the contribution is $p(s)(u(s)v'(s) - u'(s^-)v(s)) - p(s)(u(s)v'(s) - u'(s^+)v(s)) = p(s)v(s)(u'(s^+) - u'(s^-)) = -v(s)$. At $x = t$, u is smooth and v has the slope jump $-1/p(t)$; the contribution is $p(t)u(t)(v'(t^-) - v'(t^+)) = u(t)$. Adding, $u(t) - v(s) = 0$, that is, $G(t, s) = G(s, t)$.

Exercise 10.9. This is (10.10) with $L = -d^2/dx^2$, $r(x) = x$ and $f = 0$, so Theorem 10.7 gives $u(x) = \lambda \int_0^1 G(x, t) t u(t) dt$. The kernel $G(x, t)$ is not symmetric. Since $r > 0$ on $(0, 1]$, multiply by $\sqrt{x}$ and set $v = \sqrt{x} u$: then $v(x) = \lambda \int_0^1 \sqrt{x} G(x, t) \sqrt{t} v(t) dt$, and the kernel $\sqrt{xt} G(x, t)$ is symmetric. (The weight vanishes at $x = 0$, but the kernel is still continuous, so nothing breaks.)

Exercise 10.10. By Theorem 10.8 with $\varphi = 1$, a solution requires $\int_0^1 (x - c) dx = \frac{1}{2} - c = 0$, so $c = \frac{1}{2}$. For that value, integrate: $-u' = \frac{1}{2}x^2 - \frac{1}{2}x + A$, and $u'(0) = 0$ gives $A = 0$; then $u'(1) = -\frac{1}{2} + \frac{1}{2} = 0$ holds automatically, as the compatibility condition promised. Integrating again, $u = -\frac{1}{6}x^3 + \frac{1}{4}x^2 + C$ with C arbitrary.

Proposition 10.9 picks the solution with $\int_0^1 u = 0$. Since $\int_0^1 (-\frac{1}{6}x^3 + \frac{1}{4}x^2) dx = -\frac{1}{24} + \frac{1}{12} = \frac{1}{24}$, that is $C = -\frac{1}{24}$. For any other c , integrating the equation from 0 to 1 gives $u'(0) - u'(1) = \frac{1}{2} - c \neq 0$, so the two boundary conditions cannot both hold.

Exercise 10.11. Integrating $-u'' = f$ over $[0, 1]$ gives $\int_0^1 f = u'(0) - u'(1)$, which is zero under the periodic conditions. So $\int_0^1 f = 0$ is necessary, just as for the insulated rod; the constants again solve the homogeneous problem. For $f = \cos(2\pi x)$, which has mean zero, try $u = A \cos(2\pi x)$: $-u'' = 4\pi^2 A \cos(2\pi x)$, so $A = 1/(4\pi^2)$. This function is periodic with periodic derivative, and adding a constant keeps it a solution. Any two solutions differ by a solution of $u'' = 0$ with periodic conditions, which must be constant, so all solutions are $\cos(2\pi x)/(4\pi^2) + C$.

Exercise 10.12. Let $Lu = 0$ with $u(a) = u(b) = 0$. Multiply by u and integrate by parts:

$$0 = \int_a^b u(-(pu')' + qu) dx = -[p u u']_a^b + \int_a^b (p u'^2 + q u^2) dx = \int_a^b (p u'^2 + q u^2) dx,$$

since the boundary term vanishes. Both terms in the last integrand are nonnegative ($p > 0$, $q \geq 0$) and continuous, so each is identically zero; in particular $u' \equiv 0$, so u is constant, and $u(a) = 0$ makes it zero. The hypothesis of Theorem 10.4 holds, so the Green's function exists. The elastic foundation of Example 10.4 is the case $q = k^2$.

Exercise 10.13. The computed errors $|T_N - u(\frac{1}{3})|$ are 1.72×10^{-3} , 4.22×10^{-4} , 1.06×10^{-4} and 2.65×10^{-5} , with ratios 4.06, 3.97 and 4.02. On every panel that does not contain $t = \frac{1}{3}$ the integrand is smooth and the trapezoidal error is $\mathcal{O}(h^3)$ per panel, $\mathcal{O}(h^2)$ in total. On the single panel that contains the corner, the linear interpolant misses a kinked function by an amount proportional to h times the jump in slope, over a length h , so that panel

contributes $\mathcal{O}(h^2)$ as well. Both effects are second order, which is why the rate survives; the constant in front of h^2 can vary slightly with the position of the corner inside its panel. Chapter 18 studies quadrature for kernels like this one inside the Nyström method.

Exercise 10.14. Suppose G satisfies (i) to (iv). The calculation in part (c) of the proof of Theorem 10.4 can be repeated using only these properties: split $u(x) = \int_a^x G(x, t)f(t) dt + \int_x^b G(x, t)f(t) dt$, differentiate twice and use continuity of G at $x = t$, $L_x G = 0$ off the diagonal and the jump (10.6). The terms from the variable limits give exactly $-f(x)$ in $(pu')'$, so $u = \int Gf$ solves $Lu = f$, and (iii) gives the boundary conditions. Take $f = \varphi$. Then Theorem 10.8 requires $\int_a^b \varphi \cdot \varphi dx = 0$, so $\varphi \equiv 0$, a contradiction. Hence no Green's function exists whenever the homogeneous problem has a nonzero solution. Together with Theorem 10.4, this shows that a Green's function exists if and only if the homogeneous problem has only the zero solution.

Exercise 10.15. Off the diagonal $\partial_x^2 \tilde{G} = 1$, so each piece is $\frac{1}{2}x^2 + \alpha x + \beta$. The condition $\partial_x \tilde{G}(0, t) = 0$ forces $\alpha = 0$ on the left piece, and $\partial_x \tilde{G}(1, t) = 0$ forces $\alpha = -1$ on the right piece. So $\tilde{G} = \frac{1}{2}x^2 + b_1$ for $x < t$ and $\frac{1}{2}x^2 - x + b_2$ for $x > t$. The slopes at $x = t$ are t and $t - 1$, whose difference is -1 whatever t is: the jump condition is satisfied automatically. That is the compatibility condition at work, since the source (a unit point load) and the sink (a uniform removal of total strength 1) balance exactly. Continuity at $x = t$ gives $\frac{1}{2}t^2 + b_1 = \frac{1}{2}t^2 - t + b_2$, so $b_2 = b_1 + t$, and then $\tilde{G} = \frac{1}{2}x^2 - \max(x, t) + t + b_1$ in both regions. For symmetry the remaining t -dependence must make the expression symmetric, which forces $t + b_1 = \frac{1}{2}t^2 + c$ for a constant c , giving $\tilde{G} = \frac{1}{2}(x^2 + t^2) - \max(x, t) + c$. Finally $\int_0^1 \tilde{G} dx = \frac{1}{6} + \frac{1}{2}t^2 - \left(\frac{1}{2} + \frac{1}{2}t^2\right) + c = c - \frac{1}{3}$, so $c = \frac{1}{3}$.

Chapter 11 Integral Operators and Function Spaces

Exercise 11.1. The maximum of $x(1-x)$ on $[0, 1]$ occurs at $x = \frac{1}{2}$, so $\|u\|_\infty = \frac{1}{4}$. For the L^2 norm,

$$\|u\|_2^2 = \int_0^1 x^2(1-x)^2 dx = \int_0^1 (x^2 - 2x^3 + x^4) dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}.$$

So $\|u\|_2 = 1/\sqrt{30} \approx 0.1826$. With $b-a = 1$, the inequality $\|u\|_2 \leq \sqrt{b-a} \|u\|_\infty$ reads $0.1826 \leq 0.25$, which holds.

Exercise 11.2. Take $u_n(x) = x^n$. Then $\|u_n\|_\infty = 1$ and $\|u_n\|_2 = (2n+1)^{-1/2}$, so $\|u_n\|_\infty / \|u_n\|_2 = \sqrt{2n+1}$. If a constant C existed, we would have $\sqrt{2n+1} \leq C$ for all n , which is false. For the second part: the same sequence satisfies $\|u_n - 0\|_2 \rightarrow 0$ while $\|u_n - 0\|_\infty = 1$ for all n , so it converges to 0 in one norm and not in the other. (If the norms were equivalent, with constants in both directions, convergence in one would imply convergence in the other.)

Exercise 11.3. *Sup norm.* By (11.6), $\|T\|_\infty = \max_x \int_0^1 |x-t| dt = \max_x \frac{1}{2}(x^2 + (1-x)^2)$. The quadratic $\frac{1}{2}(x^2 + (1-x)^2)$ is convex with value $\frac{1}{2}$ at both endpoints and $\frac{1}{4}$ at $x = \frac{1}{2}$, so $\|T\|_\infty = \frac{1}{2}$.

Hilbert-Schmidt norm. $\int_0^1 \int_0^1 (x-t)^2 dt dx = \frac{1}{3} - 2 \cdot \frac{1}{4} + \frac{1}{3} = \frac{1}{6}$, so $\|K\|_{\text{HS}} = 1/\sqrt{6}$.

L^2 norm. We have $(Tu)(x) = x \int_0^1 u - \int_0^1 tu(t) dt$, which lies in $\text{span}\{1, x\}$. An orthonormal basis of this span is $e_1 = 1$, $e_2 = \sqrt{12}(x - \frac{1}{2})$. Compute $Te_1 = x - \frac{1}{2} = e_2/\sqrt{12}$ and

$$Te_2 = \sqrt{12}(Tx - \frac{1}{2}T1) = \sqrt{12}\left(\left(\frac{x}{2} - \frac{1}{3}\right) - \frac{1}{2}\left(x - \frac{1}{2}\right)\right) = -\frac{\sqrt{12}}{12} = -\frac{1}{\sqrt{12}}e_1.$$

Every u splits as $u = \alpha e_1 + \beta e_2 + w$ with w orthogonal to both, and $Tw = 0$ because $Tw = x\langle w, 1 \rangle - \langle w, t \rangle$ and both inner products vanish. So $Tu = \frac{1}{\sqrt{12}}(\alpha e_2 - \beta e_1)$ and $\|Tu\|_2 = \frac{1}{\sqrt{12}}(|\alpha|^2 + |\beta|^2)^{1/2} \leq \frac{1}{\sqrt{12}}\|u\|_2$, with equality for $u = e_1$. Hence $\|T\|_2 = 1/\sqrt{12} = 1/(2\sqrt{3})$. As a check, the squares of the two singular values add up to $\frac{1}{12} + \frac{1}{12} = \frac{1}{6} = \|K\|_{\text{HS}}^2$.

Exercise 11.4. For $n = 1$ the kernel is 1 on $t < x$. Suppose V^n has kernel $k_n(x, t) = (x - t)^{n-1}/(n-1)!$ on $t < x$. Then

$$\begin{aligned}(V^{n+1}u)(x) &= \int_0^x (V^n u)(s) ds = \int_0^x \int_0^s \frac{(s-t)^{n-1}}{(n-1)!} u(t) dt ds \\ &= \int_0^x \left(\int_t^x \frac{(s-t)^{n-1}}{(n-1)!} ds \right) u(t) dt,\end{aligned}$$

after exchanging the order of integration over the triangle $0 \leq t \leq s \leq x$. The inner integral is $(x-t)^n/n!$, which completes the induction. Since the kernel is nonnegative, (11.6) gives $\|V^n\|_\infty = \max_x \int_0^x (x-t)^{n-1}/(n-1)! dt = \max_x x^n/n! = 1/n!$. Hence $\|\lambda^n V^n f\|_\infty \leq |\lambda|^n \|f\|_\infty/n!$, and the Neumann series converges in $C[0, 1]$ for every λ , with sum bounded by $e^{|\lambda|} \|f\|_\infty$. This is the factorial mechanism of Chapter 7: $\|V\|_\infty = 1$ alone would only guarantee $|\lambda| < 1$.

Exercise 11.5. By Example 11.6, $(V^*Vu)(x) = \int_x^1 (Vu)(s) ds = \int_x^1 \int_0^s u(t) dt ds$. The region of integration is $x \leq s \leq 1$, $0 \leq t \leq s$; for fixed t the variable s runs from $\max(x, t)$ to 1, so the kernel is $1 - \max(x, t)$. For $u(x) = \cos(\pi x/2)$,

$$\begin{aligned}(Vu)(s) &= \frac{2}{\pi} \sin \frac{\pi s}{2}, \\ (V^*Vu)(x) &= \frac{2}{\pi} \int_x^1 \sin \frac{\pi s}{2} ds = \frac{4}{\pi^2} \left(\cos \frac{\pi x}{2} - \cos \frac{\pi}{2} \right) = \frac{4}{\pi^2} \cos \frac{\pi x}{2}.\end{aligned}$$

Then $\|Vu\|_2^2 = \langle Vu, Vu \rangle = \langle V^*Vu, u \rangle = \frac{4}{\pi^2} \|u\|_2^2$, so $\|V\|_2 \geq \|Vu\|_2/\|u\|_2 = 2/\pi$. (The value $2/\pi$ is in fact the norm; Table 11.3 confirms it numerically and Chapter 13 supplies the reason.)

Exercise 11.6. The kernel is $K(x, t) = xt^2$ for $t < x$ and 0 for $t > x$. By Theorem 11.16, $K^*(x, t) = K(t, x) = tx^2$ for $x < t$ and 0 otherwise, so $(T^*v)(x) = x^2 \int_x^1 t v(t) dt$. With $u = v = 1$: $(Tu)(x) = x \cdot x^3/3 = x^4/3$ and $\langle Tu, 1 \rangle = \int_0^1 x^4/3 dx = \frac{1}{15}$. Also $(T^*1)(x) = x^2(1 - x^2)/2$ and $\langle 1, T^*1 \rangle = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{1}{15}$.

Exercise 11.7. Let (u_n) be bounded, $\|u_n\| \leq M$. Since T is compact, there is a subsequence with $Tu_{n_k} \rightarrow y$. Since S is bounded, hence continuous, $STu_{n_k} \rightarrow Sy$; so ST is compact. For TS : $\|Su_n\| \leq \|S\|M$, so (Su_n) is a bounded sequence, and compactness of T gives a subsequence along which TSu_{n_k} converges; so TS is compact.

Exercise 11.8. Suppose $\|Te_n\| \not\rightarrow 0$. Then there are $\varepsilon > 0$ and a subsequence with $\|Te_{n_k}\| \geq \varepsilon$ for all k . The sequence (e_{n_k}) is bounded, so by compactness a further subsequence, still written (e_{n_k}) , has $Te_{n_k} \rightarrow y$ for some $y \in H$. Then $\|y\| = \lim \|Te_{n_k}\| \geq \varepsilon > 0$.

Now compute $\langle Te_{n_k}, y \rangle$ in two ways. On one hand, $\langle Te_{n_k}, y \rangle \rightarrow \langle y, y \rangle = \|y\|^2$, because $|\langle Te_{n_k}, y \rangle - \langle y, y \rangle| \leq \|Te_{n_k} - y\| \|y\| \rightarrow 0$. On the other hand, $\langle Te_{n_k}, y \rangle = \langle e_{n_k}, T^*y \rangle = \overline{\langle T^*y, e_{n_k} \rangle}$, and Bessel's inequality applied to $w = T^*y$ shows that $\langle T^*y, e_n \rangle \rightarrow 0$. So $\|y\|^2 = 0$, contradicting $\|y\| \geq \varepsilon$. Hence $\|Te_n\| \rightarrow 0$.

For the Green operator this is visible directly: $\|Te_n\|_2 = 1/(n\pi)^2$.

Exercise 11.9. Since $|xu(x)| \leq |u(x)|$ on $[0, 1]$, $\|Mu\|_2 \leq \|u\|_2$ and $\|M\|_2 \leq 1$. Let u_n equal $\sqrt{n}$ on $[1 - 1/n, 1]$ and 0 elsewhere, so $\|u_n\|_2 = 1$. Then $\|Mu_n\|_2^2 = n \int_{1-1/n}^1 x^2 dx \geq (1 - 1/n)^2$, so $\|M\|_2 \geq 1 - 1/n$ for every n , and $\|M\|_2 = 1$.

For $e_n = \sqrt{2} \sin(n\pi x)$,

$$\|Me_n\|_2^2 = \int_0^1 2x^2 \sin^2(n\pi x) dx = \int_0^1 x^2 (1 - \cos(2n\pi x)) dx = \frac{1}{3} - \frac{1}{2n^2\pi^2},$$

which tends to $\frac{1}{3}$, not 0. By Exercise 11.8, M is not compact. (Multiplication operators have no smoothing effect, and that is exactly what compactness measures.)

Exercise 11.10. The argument transfers a property of every matrix A_N to the limit operator. No such transfer is valid: the smallest eigenvalue of A_N is $1/(4N^2) \rightarrow 0$, so $\|A_N^{-1}\| \rightarrow \infty$ and the inverses have no limit. Theorem 11.14 shows that T , being compact, has no bounded inverse at all.

The range of T can be tested directly. Since $G(0, t) = \min(0, t) - 0 \cdot t = 0$ for every t , we get $(Tu)(0) = \int_0^1 G(0, t)u(t) dt = 0$ for every u ; likewise $(Tu)(1) = 0$. So every function in the range of T vanishes at both endpoints, and $f = 1$ is not in the range. (More is true: by Benchmark B3, $w = Tu$ solves $-w'' = u$ with $w(0) = w(1) = 0$, so the range consists of functions with two derivatives that vanish at both ends.)

Exercise 11.11. By Cauchy-Schwarz in t ,

$$\begin{aligned} |(Tu)(x) - (Tu)(y)| &= \left| \int_a^b (K(x, t) - K(y, t))u(t) dt \right| \\ &\leq \left(\int_a^b |K(x, t) - K(y, t)|^2 dt \right)^{1/2} \|u\|_2, \end{aligned}$$

and the first factor is at most $\sqrt{b-a} \max_t |K(x, t) - K(y, t)|$. Because K is uniformly continuous on the compact square, $\max_t |K(x, t) - K(y, t)| \leq \omega(|x - y|) \rightarrow 0$ as $y \rightarrow x$. So Tu is continuous (indeed uniformly continuous), even though u itself need only be square integrable. Continuous kernels smooth their input.

Exercise 11.12. A few lines of NumPy suffice: build $x = (\text{np.arange}(1, N+1) - 0.5)/N$, the matrix $(\text{np.minimum.outer}(x, x) - \text{np.outer}(x, x))/N$, and call $\text{np.linalg.eigvalsh}$. Sorting the eigenvalues in decreasing order and comparing with $h^2/(4 \sin^2(k\pi h/2))$, $k = 1, \dots, N$, gives agreement to about 15 significant digits for each N , so the formula is exact up to rounding. The condition number is the ratio of the $k = 1$ and $k = N$ values, $1/\sin^2(\pi h/2)$:

- $N = 8$: 26.274, against $4N^2/\pi^2 = 25.938$;
- $N = 16$: 104.087, against 103.753;
- $N = 32$: 415.345, against 415.012.

The ratio of the two columns tends to 1 because $\sin(\pi h/2) \approx \pi h/2$ for small h . Doubling N multiplies the condition number by about 4: the first-kind problem gets steadily worse to solve as the grid is refined, which is the discrete face of Theorem 11.14.

Exercise 11.13. If a subsequence x^{n_k} converged uniformly to some v , then v would be continuous (Theorem 11.3) and would equal the pointwise limit, which is 0 for $0 \leq x < 1$ and 1 at $x = 1$. That function is not continuous, a contradiction.

For the second sequence, let φ_k be the hat function of height 1 centered at $c_k = 3 \cdot 2^{-k-2}$ with support $[2^{-k-1}, 2^{-k}]$, for $k = 0, 1, 2, \dots$. The supports overlap at most at endpoints, where both hats vanish, so for $k \neq l$ the function $\varphi_k - \varphi_l$ takes the value 1 at c_k and never exceeds 1 in absolute value. Hence $\|\varphi_k - \varphi_l\|_\infty = 1$, and no subsequence is Cauchy.

Exercise 11.14. Let (u_n) be a sequence with $\|u_n\|_\infty \leq M$. Then $|(Vu_n)(x)| \leq \int_0^x |u_n| \leq M$, so (Vu_n) is uniformly bounded, and

$$|(Vu_n)(x) - (Vu_n)(y)| = \left| \int_y^x u_n(t) dt \right| \leq M|x - y|,$$

so the sequence is equicontinuous: $\rho = \varepsilon/M$ works for every n at once. By Theorem 11.13, (Vu_n) has a uniformly convergent subsequence, so V is compact on $C[0, 1]$. Theorem 11.12 needs a kernel that is continuous on the whole square, and the Volterra kernel jumps from 1 to 0 across the diagonal, so its hat-function interpolant in x does not converge uniformly near the diagonal.

Chapter 12 The Fredholm Alternative

Exercise 12.1. Write $K = a_1(x)b_1(t) + a_2(x)b_2(t)$ with $a_1 = b_1 = 1$ and $a_2 = x$, $b_2 = t$. By (12.5), $A_{ij} = \langle a_j, b_i \rangle$:

$$A = \begin{pmatrix} \langle 1, 1 \rangle & \langle x, 1 \rangle \\ \langle 1, x \rangle & \langle x, x \rangle \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

Then $\det(I - \lambda A) = (1 - \lambda)(1 - \frac{\lambda}{3}) - \frac{\lambda^2}{4} = 1 - \frac{4}{3}\lambda + \frac{1}{12}\lambda^2$. Setting this to zero gives $\lambda^2 - 16\lambda + 12 = 0$, so $\lambda = 8 \pm \sqrt{52} = 8 \pm 2\sqrt{13}$. By Theorem 12.3, for every other λ the homogeneous equation has only the zero solution and the equation is uniquely solvable for every f . At each characteristic value the null space is one-dimensional, since $I - \lambda A$ is a nonzero 2×2 matrix there.

Exercise 12.2. By Example 12.3, the compatibility condition at $\lambda = 2$ is $\int_0^1 x f(x) dx = 0$. For $f = x^2 + \gamma$ this is $\frac{1}{4} + \frac{\gamma}{2} = 0$, so $\gamma = -\frac{1}{2}$. The null space is $\text{span}\{1\}$, so the solutions are $u_0 + C$ for one solution u_0 . Try $u_0 = f = x^2 - \frac{1}{2}$: $\int_0^1 t u_0 dt = \frac{1}{4} - \frac{1}{4} = 0$, so $f + 2 \int_0^1 t u_0 = f = u_0$, and u_0 is a solution. All solutions: $u = x^2 - \frac{1}{2} + C$. Check: $\int_0^1 t(t^2 - \frac{1}{2} + C) dt = \frac{C}{2}$ and $f + 2 \cdot \frac{C}{2} = u$.

Exercise 12.3. The reader tested f against the null space of $I - 2T$, which is the right test only for a self-adjoint T . Here $K(x, t) = x$, so the adjoint kernel is $K^*(x, t) = K(t, x) = t$ and the adjoint homogeneous equation is $v(x) = 2 \int_0^1 t v(t) dt$. Its solutions are the constants: any solution is constant, and a constant $v = C$ satisfies $2 \int_0^1 t C dt = C$. By Theorem 12.2, the equation is solvable exactly when $\int_0^1 f \cdot 1 dx = 0$.

For $f = 1 - \frac{3}{2}x$, $\int_0^1 f = 1 - \frac{3}{4} = \frac{1}{4} \neq 0$, so there is no solution. A direct check confirms it: put $c = \int_0^1 u$. Then $u = f + 2cx$ and $c = \int_0^1 f + 2c \int_0^1 x = \frac{1}{4} + c$, which is impossible. This is Example 12.3 with the roles of the two null spaces swapped.

Exercise 12.4. If $T^* = T$ and λ is real, then $I - \bar{\lambda}T^* = I - \lambda T$, so the adjoint null space in (12.3) is $\mathcal{N}(I - \lambda T)$ itself.

For nonreal λ there is nothing to fail. If $u - \lambda Tu = 0$ with $u \neq 0$, then $\langle u, u \rangle = \lambda \langle Tu, u \rangle$, and $\langle Tu, u \rangle = \langle u, Tu \rangle = \overline{\langle Tu, u \rangle}$ is real. Since $\langle u, u \rangle > 0$, the number $\langle Tu, u \rangle$ is nonzero and $\lambda = \langle u, u \rangle / \langle Tu, u \rangle$ is real. So a self-adjoint operator has only real characteristic values, a nonreal λ always falls under case (i), and no compatibility condition is needed. Chapter 13 develops this into the spectral theory of symmetric kernels.

Exercise 12.5. Let $N = \mathcal{N}(I - \lambda T)$, which is finite-dimensional, with orthonormal basis $e_1, \dots, e_m$, and let $Pw = \sum \langle w, e_i \rangle e_i$. The solutions are $u_0 + w$ with $w \in N$. The vector $u^\dagger = u_0 - Pu_0$ is a solution (since $Pu_0 \in N$), and it is orthogonal to every e_i , hence to N . If two solutions $u^\dagger$ and $u^\dagger + w$ are both orthogonal to N , then $w \in N$ is orthogonal to itself, so $w = 0$: the orthogonal solution is unique. For any other solution $u = u^\dagger + w$ with $0 \neq w \in N$, Pythagoras gives $\|u\|^2 = \|u^\dagger\|^2 + \|w\|^2 > \|u^\dagger\|^2$.

For Benchmark B2 at $\lambda = 3$ with $f = x - \frac{2}{3}$, the solutions are $x - \frac{2}{3} + Cx$ and $N = \text{span}\{x\}$. Orthogonality requires $\int_0^1 x(x - \frac{2}{3} + Cx) dx = \frac{C}{3} = 0$, so $u^\dagger = x - \frac{2}{3}$, as found in Example 12.2.

Exercise 12.6. (a) Split the integral at $t = x$:

$$\begin{aligned} \int_0^1 G_1(x, t) dt &= \frac{\cosh(1-x) \int_0^x \cosh t dt + \cosh x \int_x^1 \cosh(1-t) dt}{\sinh 1} \\ &= \frac{\cosh(1-x) \sinh x + \cosh x \sinh(1-x)}{\sinh 1}. \end{aligned}$$

By the addition formula the numerator is $\sinh(x + 1 - x) = \sinh 1$, so the integral is 1.

(b) The condition $\int_0^1 \cos \pi x dx = 0$ holds. The function $u = \cos(\pi x)/\pi^2$ satisfies $-u'' = \cos \pi x$ and $u'(x) = -\sin(\pi x)/\pi$ vanishes at 0 and 1. The null space consists of constants, so all solutions are $u = \cos(\pi x)/\pi^2 + C$.

(c) $\int_0^1 x dx = \frac{1}{2} \neq 0$, so $f = x$ is not admissible. Directly: integrating $-u'' = x$ gives $u'(0) - u'(1) = \frac{1}{2}$, contradicting the boundary conditions.

Exercise 12.7. (a) $Tx = \cos 2\pi x \int_0^1 t \cos 2\pi t dt + \sin 2\pi x \int_0^1 t \sin 2\pi t dt = 0 + \sin 2\pi x \cdot (-\frac{1}{2\pi})$. By Example 12.4, $u = f + \frac{\lambda}{1-\lambda/2} Tf$, which at $\lambda = 1$ is $u = x + 2Tx = x - \sin(2\pi x)/\pi$. Check: $T \sin 2\pi x = \frac{1}{2} \sin 2\pi x$, so $Tu = -\frac{\sin 2\pi x}{2\pi} - \frac{\sin 2\pi x}{2\pi} = -\frac{\sin 2\pi x}{\pi}$ and $x + Tu = u$.

(b) At $\lambda = 2$ the compatibility condition requires $f = x$ to be orthogonal to $\cos 2\pi x$ and $\sin 2\pi x$. The first inner product is 0, but $\int_0^1 x \sin 2\pi x dx = -\frac{1}{2\pi} \neq 0$. No solution exists.

Exercise 12.8. Since $\|x\|_2^2 = \int_0^1 x^2 = \frac{1}{3}$, the orthogonal projection onto $\text{span}\{x\}$ is $Pu = \langle u, x \rangle x / \|x\|_2^2 = 3x\langle u, x \rangle$. So $Tu = x\langle u, x \rangle = \frac{1}{3}Pu$. Using $P^2 = P$,

$$(I - \lambda T)\left(I + \frac{\lambda}{3 - \lambda}P\right) = I + \frac{\lambda}{3 - \lambda}P - \frac{\lambda}{3}P - \frac{\lambda^2}{3(3 - \lambda)}P = I + \lambda P \frac{3 - (3 - \lambda) - \lambda}{3(3 - \lambda)} = I,$$

and the product in the other order is also I because all the operators involved commute. Every u splits as $u = Pu + (I - P)u$ with orthogonal pieces, and

$$(I - \lambda T)^{-1}u = \frac{3}{3 - \lambda}Pu + (I - P)u.$$

So $\|(I - \lambda T)^{-1}u\|_2^2 = \frac{9}{|3 - \lambda|^2}\|Pu\|_2^2 + \|(I - P)u\|_2^2 \leq \max\left(1, \frac{9}{|3 - \lambda|^2}\right)\|u\|_2^2$, with equality for $u = x$ or for u orthogonal to x . Hence $\|(I - \lambda T)^{-1}\|_2 = \max(1, 3/|3 - \lambda|)$. This is the solid curve in Figure 12.3. At $\lambda = 2.99$ the norm is 300: the problem is uniquely solvable but amplifies data errors by a factor of 300.

Exercise 12.9. The equation $u - \lambda Mu = f$ is $(1 - \lambda x)u(x) = f(x)$. If $\lambda \notin [1, \infty)$, the continuous function $1 - \lambda x$ has no zero on $[0, 1]$ (a zero would require $\lambda = 1/x \geq 1$), so $\delta = \min_{[0, 1]} |1 - \lambda x| > 0$. Then $u = f/(1 - \lambda x)$ is the unique solution and $\|u\|_2 \leq \|f\|_2/\delta$, so the inverse is bounded.

If $\lambda \geq 1$ is real, then $x_0 = 1/\lambda \in (0, 1]$. The homogeneous equation $(1 - \lambda x)u = 0$ forces $u = 0$ except at the single point x_0 , so $I - \lambda M$ is injective. For $f = 1$ the only candidate is $u = 1/(1 - \lambda x)$, and near x_0 we have $|u|^2 = 1/(\lambda^2(x - x_0)^2)$, whose integral diverges on either side of x_0 inside $[0, 1]$. So no solution exists in $L^2(0, 1)$ and $I - \lambda M$ is not onto.

The bad set $[1, \infty)$ is an entire interval, whereas for a compact operator the bad values are isolated and have finite multiplicity. There is no contradiction because M is not compact (Chapter 11), and none of these λ are characteristic values: no nonzero homogeneous solution exists at all.

Exercise 12.10. The Volterra kernel $K(x, t)$ for $t \leq x$, extended by 0 for $t > x$, is bounded and so Hilbert–Schmidt; by Chapter 11 the operator is compact on $L^2(a, b)$. Since the homogeneous equation has only the zero solution for every λ , case (i) of Theorem 12.2 applies for every λ : the equation has exactly one solution for each $f \in L^2(a, b)$, and $\|u\|_2 \leq C(\lambda)\|f\|_2$. In particular a Volterra operator with continuous kernel has no characteristic values at all.

The alternative is silent about how to find u and about the size of $C(\lambda)$. Those come from the resolvent kernel $R(x, t; \lambda) = \sum \lambda^{n-1} K_n(x, t)$ and the factorial estimates of Chapter 7, which give both the formula $u = f + \lambda \int_a^x R(x, t; \lambda) f(t) dt$ and an explicit bound. For continuous f , Proposition 12.6 shows the solution is continuous.

Exercise 12.11. (a) The trapezoid rule integrates linear functions exactly, so $\sum_j w_j t_j = \int_0^1 t \, dt = \frac{1}{2}$. For $g(t) = t^2$ the trapezoid error on a uniform grid is exactly $\frac{h^2}{12}(g'(1) - g'(0)) = \frac{h^2}{12} \cdot 2 = \frac{h^2}{6}$, because g'' is constant; so $\sum_j w_j t_j^2 = \frac{1}{3} + \frac{h^2}{6}$.

(b) The Nyström equations are $u_i = 1 + 3t_i \sum_j w_j t_j u_j$. With $c = \sum_j w_j t_j u_j$, every $u_i = 1 + 3t_i c$, and substituting back gives

$$c = \sum_j w_j t_j (1 + 3t_j c) = \frac{1}{2} + 3c \left(\frac{1}{3} + \frac{h^2}{6} \right) = \frac{1}{2} + c + \frac{h^2}{2} c,$$

so $c = -1/h^2$. The discrete system is uniquely solvable because its characteristic value $\lambda_N = 1/\sum_j w_j t_j^2 = 3/(1 + h^2/2)$ differs from 3. The Nyström interpolant is $u_N(x) = 1 + 3xc = 1 - 3x/h^2$.

(c) For $N = 10, 20, 40$ the solver returns $u_N(1) = 1 - 3N^2 = -299, -1199, -4799$ without complaint. The values grow like N^2 as the grid is refined, which is the numerical signature of a continuous problem with no solution: the discrete problem sits a distance of order h^2 from resonance, and its solution is of order $1/h^2$. The right conclusion is to test the compatibility condition $\int_0^1 x f(x) \, dx = \frac{1}{2} \neq 0$ before trusting the numbers, and Chapter 18 returns to this failure mode.

Exercise 12.12. Suppose $E = \mathcal{N}(T - \mu I)$ is infinite-dimensional. Applying Gram–Schmidt to a sequence of linearly independent vectors in E gives an orthonormal sequence (e_n) in E . Each e_n satisfies $Te_n = \mu e_n$, so

$$\|Te_n - Te_m\| = |\mu| \|e_n - e_m\| = |\mu| \sqrt{\|e_n\|^2 + \|e_m\|^2} = |\mu| \sqrt{2}$$

for $n \neq m$. The sequence (e_n) is bounded, but no subsequence of (Te_n) is Cauchy, since its terms stay $|\mu| \sqrt{2} > 0$ apart. This contradicts the compactness of T . Hence E is finite-dimensional. (With $\mu = 1/\lambda$, $E = \mathcal{N}(I - \lambda T)$, which is the dimension statement in Theorem 12.2(ii)(a).)

Exercise 12.13. The adjoint null space is spanned by the constant $v_1 = 1$, which already has unit norm on $[0, 1]$. The inconsistency is $\langle f, 1 \rangle = \int_0^1 e^x \, dx = e - 1 \approx 1.718$, the net heat input, and the projected data are $f_c = e^x - (e - 1)$. Integrating $-u'' = e^x - (e - 1)$ twice gives $u = -e^x + \frac{e-1}{2}x^2 + Ax + B$. The condition $u'(0) = -1 + A = 0$ gives $A = 1$, and then $u'(1) = -e + (e - 1) + 1 = 0$ holds automatically, as it must for compatible data. The null space is the constants, and the mean-zero normalization gives

$$0 = \int_0^1 u = -(e - 1) + \frac{e - 1}{6} + \frac{1}{2} + B, \quad B = \frac{5(e - 1)}{6} - \frac{1}{2}.$$

So $u = -e^x + \frac{e-1}{2}x^2 + x + \frac{5(e-1)}{6} - \frac{1}{2}$. Here the inconsistency is as large as the data themselves, so in a real measurement it would signal that the rod is not insulated, or that the steady-state model is wrong, rather than a small error to be projected away.

Chapter 13 Symmetric Kernels and Spectral Methods

Exercise 13.1. All 5 kernels are real, so self-adjointness is the same as symmetry $K(x, t) = K(t, x)$.

- (a) $e^{x+t} = e^{t+x}$: symmetric.
- (b) e^{x-t} becomes e^{t-x} after the swap, which differs whenever $x \neq t$: not symmetric.
- (c) $|x - t| = |t - x|$: symmetric.
- (d) xt^2 becomes tx^2 : not symmetric.
- (e) $\cos(\pi(x - t)) = \cos(\pi(t - x))$ because cosine is even: symmetric.

By Proposition 13.2 the kernels (a), (c) and (e) give self-adjoint operators. For (b) and (d) the adjoint kernel $K(t, x)$ differs from K on a set of positive area, so $T^* \neq T$.

Exercise 13.2. Expand $\cos(\pi(x - t)) = \cos \pi x \cos \pi t + \sin \pi x \sin \pi t$, a symmetric kernel of rank 2. On $[0, 1]$,

$$\int_0^1 \cos^2 \pi t \, dt = \int_0^1 \sin^2 \pi t \, dt = \frac{1}{2}, \quad \int_0^1 \sin \pi t \cos \pi t \, dt = 0.$$

Hence $T \cos \pi x = \frac{1}{2} \cos \pi x$ and $T \sin \pi x = \frac{1}{2} \sin \pi x$. So $\mu = \frac{1}{2}$ has the 2-dimensional eigenspace spanned by $\cos \pi x$ and $\sin \pi x$, which are orthogonal; normalized, they are $\sqrt{2} \cos \pi x$ and $\sqrt{2} \sin \pi x$. Any function orthogonal to both is mapped to 0, and since the range of T lies in their span, there are no other nonzero eigenvalues. The characteristic value is $\lambda = 2$, of multiplicity 2.

Parseval: $\sum \mu_n^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, and $\|K\|_{\text{HS}}^2 = \int_0^1 \int_0^1 \cos^2(\pi(x - t)) \, dt \, dx = \frac{1}{2}$, since for each fixed x the inner integral is the average of $\cos^2$ over a full half-period shift, which is $\frac{1}{2}$.

Exercise 13.3. Hermitian: $\overline{K(t, x)} = \overline{i(t - x)} = -i(t - x) = i(x - t) = K(x, t)$.

For $x - t$, the range lies in $\text{span}\{1, x\}$. With $\varphi = \alpha + \beta x$,

$$\int_0^1 (x - t)(\alpha + \beta t) \, dt = x\left(\alpha + \frac{\beta}{2}\right) - \left(\frac{\alpha}{2} + \frac{\beta}{3}\right),$$

so $\nu\alpha = -\frac{\alpha}{2} - \frac{\beta}{3}$ and $\nu\beta = \alpha + \frac{\beta}{2}$. The determinant condition is $(-\frac{1}{2} - \nu)(\frac{1}{2} - \nu) + \frac{1}{3} = \nu^2 + \frac{1}{12} = 0$, so $\nu = \pm i/(2\sqrt{3}) = \pm i\sqrt{3}/6$: purely imaginary, as expected for a real antisymmetric kernel.

The kernel $i(x-t)$ is i times $x-t$, so its eigenvalues are $i\nu = \mp\sqrt{3}/6$, real as Theorem 13.5 requires. Check with Parseval: $\|K\|_{\text{HS}}^2 = \int_0^1 \int_0^1 (x-t)^2 = \frac{1}{6}$ and $2 \cdot \frac{1}{12} = \frac{1}{6}$.

Exercise 13.4. By Theorem 13.5, μ is real. Since K is real, conjugating

$$\int K(x, t)\varphi(t) dt = \mu\varphi(x)$$

gives $\int K(x, t)\overline{\varphi(t)} dt = \overline{\mu\varphi(x)}$. So $T\overline{\varphi} = \mu\overline{\varphi}$. By linearity,

$$T(\text{Re } \varphi) = T\left(\frac{\varphi + \overline{\varphi}}{2}\right) = \mu \text{Re } \varphi, \quad T(\text{Im } \varphi) = T\left(\frac{\varphi - \overline{\varphi}}{2i}\right) = \mu \text{Im } \varphi.$$

At least one of $\text{Re } \varphi$, $\text{Im } \varphi$ is nonzero, so it is a real eigenfunction. For a whole eigenspace E (finite dimensional when $\mu \neq 0$), the real and imaginary parts of a basis of E are real functions in E that span E over $\mathbb{C}$. Extract a linearly independent subset and apply Gram-Schmidt; real arithmetic keeps the functions real.

Exercise 13.5. Since $1 - \max(x, t) = \min(1 - x, 1 - t)$, the change of variables $y = 1 - x$, $s = 1 - t$ turns $\int_0^1 (1 - \max(x, t))\varphi(t) dt = \mu\varphi(x)$ into $\int_0^1 \min(y, s)\psi(s) ds = \mu\psi(y)$ with $\psi(y) = \varphi(1 - y)$. By Example 13.4, $\mu_n = 1/((n - \frac{1}{2})\pi)^2$ and $\psi_n(y) = \sqrt{2} \sin((n - \frac{1}{2})\pi y)$. Returning to x ,

$$\varphi_n(x) = \sqrt{2} \sin\left((n - \frac{1}{2})\pi(1 - x)\right) = (-1)^{n+1} \sqrt{2} \cos\left((n - \frac{1}{2})\pi x\right),$$

and the sign can be dropped. Direct check for $n = 1$: differentiating $T\varphi = \mu\varphi$ as in Example 13.4 gives $\mu\varphi'' = -\varphi$ with $\varphi'(0) = 0$ and $\varphi(1) = 0$, which $\cos(\pi x/2)$ satisfies with $\mu = 4/\pi^2$.

Exercise 13.6. The reader's formula expands only the part of $f = x^2$ in $\text{span}\{1, x\}$, but $x^2 = f_0 + (\text{projection onto } \text{span}\{1, x\})$ with $f_0 \neq 0$ in the null space of T . Every solution of $u = x^2 + Tu$ has the form $x^2 + (\text{something in the range of } T)$, so it is x^2 plus a linear polynomial, never a linear polynomial alone.

Correct solution: $\lambda = 1$ is not a characteristic value ($4\sqrt{3} - 6 \approx 0.928$ and -12.93 are), so a unique solution exists. Write $u = x^2 + Ax + B$ with $A = \int_0^1 u$ and $B = \int_0^1 tu$:

$$A = \frac{1}{3} + \frac{A}{2} + B, \quad B = \frac{1}{4} + \frac{A}{3} + \frac{B}{2}.$$

The first gives $A = \frac{2}{3} + 2B$; substituting in the second gives $-\frac{B}{6} = \frac{1}{3} + \frac{2}{9} = \frac{17}{36}$, so $B = -\frac{17}{6}$ and $A = -5$. Hence $u(x) = x^2 - 5x - \frac{17}{6}$. Check: $\int_0^1 u = \frac{1}{3} - \frac{5}{2} - \frac{17}{6} = -5 = A$ and $\int_0^1 tu = \frac{1}{4} - \frac{5}{3} - \frac{17}{12} = -\frac{17}{6} = B$. Formula (13.8) gives the same answer, because it starts from f itself.

Exercise 13.7. *Closed form.* As in Example 13.5, $w = u - 1 = \lambda Gu$ vanishes at both ends and $-w'' = \lambda u$. So $-u'' = \lambda u$ with $u(0) = u(1) = 1$. For $\lambda = k^2 > 0$, the solution symmetric about $x = \frac{1}{2}$ is

$$u(x) = \frac{\cos(k(x - \frac{1}{2}))}{\cos(k/2)},$$

valid when $\cos(k/2) \neq 0$; for $\lambda = -k^2 < 0$ it is $\cosh(k(x - \frac{1}{2}))/\cosh(k/2)$, and for $\lambda = 0$ it is $u = 1$.

Classification. The characteristic values are $\lambda_n = (n\pi)^2$. The data coefficients are

$$\langle 1, \sqrt{2} \sin(n\pi x) \rangle = \sqrt{2} \frac{1 - (-1)^n}{n\pi},$$

nonzero for odd n and zero for even n . By Theorem 13.15:

- at $\lambda = ((2m - 1)\pi)^2$ there is no solution; the closed form confirms this, because $\cos(k/2) = \cos((2m - 1)\pi/2) = 0$;
- at $\lambda = (2m\pi)^2$ the data is compatible and there are infinitely many solutions. The closed form with $k = 2m\pi$ gives $\cos(2m\pi x - m\pi)/\cos(m\pi) = \cos(2m\pi x)$, and adding the homogeneous solution gives $u = \cos(2m\pi x) + C \sin(2m\pi x)$.

Check the family at $\lambda = 4\pi^2$: $-u'' = 4\pi^2 u$ and $u(0) = u(1) = 1$ for every C , so $u - 1$ vanishes at both ends and equals $G(4\pi^2 u)$.

Exercise 13.8. With $t_j = (j - \frac{1}{2})/100$ and $A_{ij} = G(t_i, t_j)/100$, a symmetric eigenvalue routine gives the 5 largest eigenvalues 0.101330, 0.025338, 0.011266, 0.006341 and 0.004061 (6 decimals). The exact values are 0.101321, 0.025330, 0.011258, 0.006333 and 0.004053. The relative errors are about 8.2×10^{-5} , 3.3×10^{-4} , 7.4×10^{-4} , 1.3×10^{-3} and 2.1×10^{-3} . The ratio of the n th error to the first is close to n^2 , which matches the explanation in the text: the error behaves like $(nh)^2$ with $h = 1/N$, because the n th eigenfunction oscillates n times faster.

Exercise 13.9. Suppose $K(x_0, x_0) = -c < 0$. By continuity there is an interval $J = [x_0 - \varepsilon, x_0 + \varepsilon] \cap [a, b]$ of positive length such that $K(x, t) < -c/2$ for all $x, t \in J$. Let $u = 1$ on J and 0 elsewhere. Then

$$\langle Tu, u \rangle = \iint_J K(x, t) dt dx < -\frac{c}{2} |J|^2 < 0,$$

which contradicts positivity. Hence $K(x, x) \geq 0$ for every x . The kernels $\min(x, t)$ and $G(x, t)$ pass this test, with diagonals x and $x - x^2$; the kernel $x + t$ passes it too, even though it is not positive, which shows that the condition is necessary but not sufficient.

Exercise 13.10. First the norm. By symmetry,

$$\int_0^1 \int_0^1 \min(x, t)^2 dt dx = 2 \int_0^1 \int_0^x t^2 dt dx = 2 \int_0^1 \frac{x^3}{3} dx = \frac{1}{6}.$$

The eigenvalues from Example 13.4 are $\mu_n = 1/((n - \frac{1}{2})\pi)^2 = 4/((2n - 1)^2\pi^2)$, and they are all of the nonzero eigenvalues. Corollary 13.11 gives

$$\sum_{n=1}^{\infty} \frac{16}{(2n-1)^4\pi^4} = \frac{1}{6}, \quad \text{so} \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}.$$

Check against Euler's sum from Example 13.3: the odd terms of $\sum 1/n^4 = \pi^4/90$ make up $1 - \frac{1}{16} = \frac{15}{16}$ of it, and $\frac{15}{16} \cdot \frac{\pi^4}{90} = \frac{\pi^4}{96}$.

Exercise 13.11. $(Tu)(x) = x c(u)$ with $c(u) = \int_0^1 (2-3t)u(t) dt$. For $u = 1$, $c = \frac{1}{2}$, so $T \neq 0$. Applying T twice,

$$(T^2u)(x) = x c(Tu) = x c(u) \int_0^1 (2-3s)s ds = x c(u)(1-1) = 0.$$

If $T\varphi = \mu\varphi$ with $\varphi \neq 0$, then $0 = T^2\varphi = \mu^2\varphi$, so $\mu = 0$. Hence T has no nonzero eigenvalue and no characteristic value: $u = f + \lambda Tu$ is uniquely solvable for every λ , with $u = f + \lambda Tf$ (check: $\lambda Tu = \lambda Tf + \lambda^2 T^2f = \lambda Tf$).

There is no contradiction with Theorem 13.10, whose hypothesis is a Hermitian kernel; $x(2-3t)$ is not symmetric. It is a finite-rank relative of the Volterra operator of Section 13.7.

Exercise 13.12. (a) For real u, v , Fubini on the triangle $0 \leq t \leq x \leq 1$ gives

$$\langle Vu, v \rangle = \int_0^1 \int_0^x u(t) dt v(x) dx = \int_0^1 u(t) \int_t^1 v(x) dx dt,$$

so $V^*v(t) = \int_t^1 v(x) dx$. (The complex case adds conjugates in the same places.)

(b) $(V^*Vu)(x) = \int_x^1 \int_0^s u(t) dt ds$. For fixed t , the variable s runs over $s \geq x$ and $s \geq t$, that is from $\max(x, t)$ to 1, so

$$(V^*Vu)(x) = \int_0^1 u(t)(1 - \max(x, t)) dt.$$

(c) For any bounded operator, $\|Vu\|_2^2 = \langle V^*Vu, u \rangle \leq \|V^*V\| \|u\|_2^2$, and $\|V^*V\| \leq \|V\|^2$; together these give $\|V\|^2 = \|V^*V\|$. The operator V^*V is self-adjoint with the continuous kernel $1 - \max(x, t)$, whose largest eigenvalue by Exercise 13.5 is $\mu_1 = 4/\pi^2$. By Lemma 13.8 and Theorem 13.10, $\|V^*V\| = |\mu_1| = 4/\pi^2$. Hence $\|V\| = 2/\pi \approx 0.6366$. The maximizing function is $u(x) = \sqrt{2} \cos(\pi x/2)$, for which $Vu = \frac{2\sqrt{2}}{\pi} \sin(\pi x/2)$ has norm exactly $2/\pi$.

Exercise 13.13. By Theorem 13.14, $u = (I - \lambda T)^{-1}f = f_0 + \sum_n \langle f, \varphi_n \rangle \varphi_n / (1 - \lambda \mu_n)$, since $f + \lambda \mu_n \langle f, \varphi_n \rangle / (1 - \lambda \mu_n)$ multiplies the n th coordinate by $1/(1 - \lambda \mu_n)$. By Pythagoras,

$$\|u\|_2^2 = \|f_0\|_2^2 + \sum_n \frac{|\langle f, \varphi_n \rangle|^2}{|1 - \lambda \mu_n|^2} \leq S^2 \left(\|f_0\|_2^2 + \sum_n |\langle f, \varphi_n \rangle|^2 \right) = S^2 \|f\|_2^2,$$

where S is the claimed maximum. So the norm is at most S . For the reverse, $f = \varphi_n$ gives $\|u\|_2 = 1/|1 - \lambda \mu_n|$, and either $\mathcal{N}(T) \neq \{0\}$ (take $f \in \mathcal{N}(T)$, giving ratio 1) or there are infinitely many $\mu_n \rightarrow 0$ (ratios tending to 1). In either case the supremum S is approached, so the norm equals S . The infinite dimension is used here: a finite list of eigenfunctions cannot span $L^2(a, b)$.

For G at $\lambda = 9$: $\mu_1 = 1/\pi^2$ gives $1/(1 - 9/\pi^2) = \pi^2/(\pi^2 - 9) \approx 11.35$, and the other modes give factors below 1.3. So a perturbation of the data along $\sin \pi x$ is amplified about 11 times in the solution. As $\lambda \rightarrow \pi^2$ this factor blows up, which is the quantitative face of resonance and the reason Chapter 14 studies the resolvent near its poles.

Chapter 14 Resolvents and Fredholm Determinants

Exercise 14.1. The kernel is rank one with $a(x) = e^x$ and $b(t) = e^t$, so $A = \int_0^1 e^{2t} dt = \frac{e^2-1}{2}$ and $D(\lambda) = 1 - \lambda \frac{e^2-1}{2}$. By Theorem 14.5, $R(x, t; \lambda) = e^x e^t / D(\lambda)$. The only characteristic value is the zero of D , $\lambda = 2/(e^2 - 1) \approx 0.3130$. Check with the resolvent equation:

$$e^{x+t} + \lambda \int_0^1 e^{x+s} \frac{e^{s+t}}{D(\lambda)} ds = e^{x+t} \left(1 + \frac{\lambda(e^2-1)/2}{D(\lambda)} \right) = \frac{e^{x+t}}{D(\lambda)}.$$

Exercise 14.2. Take $a_1 = 1, b_1 = 1, a_2 = x, b_2 = t$. On $[-1, 1]$, $\int 1 = 2, \int t = 0$ and $\int t^2 = \frac{2}{3}$, so $A = \text{diag}(2, \frac{2}{3})$ and $D(\lambda) = (1 - 2\lambda)(1 - \frac{2\lambda}{3})$. Because A is diagonal,

$$R(x, t; \lambda) = \frac{1}{1 - 2\lambda} + \frac{xt}{1 - 2\lambda/3}.$$

The characteristic values are $\frac{1}{2}$ (eigenfunction 1, eigenvalue 2) and $\frac{3}{2}$ (eigenfunction x , eigenvalue $\frac{2}{3}$). The kernel is symmetric and the functions 1 and x are orthogonal on $[-1, 1]$, so this is the expansion (14.9) with the normalized eigenfunctions $1/\sqrt{2}$ and $\sqrt{3/2} x$.

Exercise 14.3. With $R(x, t; \lambda) = 3xt/(3 - \lambda)$,

$$\int_0^1 \frac{3xs}{3 - \lambda} \cdot \frac{3st}{3 - \nu} ds = \frac{9xt}{(3 - \lambda)(3 - \nu)} \int_0^1 s^2 ds = \frac{3xt}{(3 - \lambda)(3 - \nu)}.$$

On the other side,

$$\frac{3xt}{3 - \lambda} - \frac{3xt}{3 - \nu} = 3xt \frac{(3 - \nu) - (3 - \lambda)}{(3 - \lambda)(3 - \nu)} = (\lambda - \nu) \frac{3xt}{(3 - \lambda)(3 - \nu)},$$

which is $(\lambda - \nu)$ times the integral, as (14.7) states.

Exercise 14.4. Definition 14.6 has $D(\lambda) = \sum (-\lambda)^m A_m / m!$. With $A_1 = 1$ and $A_2 = -\frac{1}{6}$ this gives $D(\lambda) = 1 - \lambda + \frac{\lambda^2}{2} \left(-\frac{1}{6}\right) = 1 - \lambda - \frac{\lambda^2}{12}$, and $A_m = 0$ for $m \geq 3$ because every 3×3 determinant of values of a rank-two kernel vanishes. The correct characteristic values are $-6 \pm 4\sqrt{3}$.

Two independent checks expose the slip. First, Example 14.3 computes $\det(I - \lambda A)$ directly and gets $1 - \lambda - \lambda^2/12$. Second, a characteristic value must produce a nonzero solution of $u = \lambda Tu$. At $\lambda = -3 + \sqrt{15} \approx 0.873$, the coefficient system of Example 14.3 has determinant $D(0.873) \approx 1 - 0.873 - 0.0635 = 0.0635 \neq 0$, so only $u = 0$ solves the homogeneous equation. The correct zero $4\sqrt{3} - 6 \approx 0.928$ also matches $1/\mu_+$ from Chapter 13.

Exercise 14.5. Write $K(x, s) = \mathbf{a}(x)^\top \mathbf{b}(s)$ and $R(s, t) = \mathbf{a}(s)^\top (I - \lambda A)^{-1} \mathbf{b}(t)$. Then

$$\int_a^b K(x, s)R(s, t) ds = \mathbf{a}(x)^\top \left(\int_a^b \mathbf{b}(s)\mathbf{a}(s)^\top ds \right) (I - \lambda A)^{-1} \mathbf{b}(t) = \mathbf{a}(x)^\top A(I - \lambda A)^{-1} \mathbf{b}(t),$$

because the matrix $\int \mathbf{b} \mathbf{a}^\top$ has entries $\int b_i a_j = A_{ij}$. Therefore

$$K + \lambda \int KR = \mathbf{a}(x)^\top [I + \lambda A(I - \lambda A)^{-1}] \mathbf{b}(t) = \mathbf{a}(x)^\top (I - \lambda A)^{-1} \mathbf{b}(t) = R,$$

since $I + \lambda A(I - \lambda A)^{-1} = [(I - \lambda A) + \lambda A](I - \lambda A)^{-1} = (I - \lambda A)^{-1}$.

Exercise 14.6. $A_1 = \int_0^1 s^2 ds = \frac{1}{3}$. For A_2 the determinant is $s_1^2 s_2^2 - (s_1 s_2)(s_2 s_1) = 0$, so $A_2 = 0$; likewise every A_m with $m \geq 2$ vanishes, because the kernel has rank one. For B_1 ,

$$\det \begin{bmatrix} xt & xs \\ st & s^2 \end{bmatrix} = xts^2 - xs \cdot st = 0,$$

so $B_1 = 0$ and all later B_m vanish too. Hence $D(\lambda) = 1 - \lambda/3$, $D(x, t; \lambda) = xt$, and $R = xt/(1 - \lambda/3) = 3xt/(3 - \lambda)$, as in Example 14.1.

Exercise 14.7. If $u = f + \lambda Tu$, then $w = u - f = \lambda Tu$. From Chapter 13, $v = Tu$ satisfies $-v'' = u$, $v(0) = 0$ and $v'(1) = 0$. So

$$-w'' - \lambda w = \lambda f, \quad w(0) = 0, \quad w'(1) = 0,$$

and $R(\cdot, \cdot; \lambda)$ is the Green's function of $-\frac{d^2}{dx^2} - \lambda$ with these conditions. With $\lambda = k^2$, the solution that satisfies the left condition is $y_L = \sin(kx)$, and the one that satisfies the right condition is $y_R = \cos(k(1 - x))$, since $y'_R(1) = 0$. Their Wronskian is

$$y_L y'_R - y'_L y_R = \sin(kx) k \sin(k(1 - x)) - k \cos(kx) \cos(k(1 - x)) = -k \cos k.$$

With the slope jump -1 of Chapter 10, $R = y_L(x_<)y_R(x_>)/(k \cos k)$, which is the stated formula. As $k \rightarrow 0$ it tends to $x_< = \min(x, t)$, as $R_0 = T$ requires. The poles are the zeros of $\cos k$, $k = (n - \frac{1}{2})\pi$, that is $\lambda = ((n - \frac{1}{2})\pi)^2 = 1/\mu_n$ with μ_n from Chapter 13.

Exercise 14.8. $A_1 = \int_0^1 \min(s, s) ds = \int_0^1 s ds = \frac{1}{2}$. For A_2 ,

$$A_2 = \iint [s_1 s_2 - \min(s_1, s_2)^2] ds_1 ds_2 = \frac{1}{4} - \|\min\|_{\text{HS}}^2 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12},$$

using $\|\min\|_{\text{HS}}^2 = \frac{1}{6}$ from Chapter 13. So $D(\lambda) = 1 - \frac{\lambda}{2} + \frac{\lambda^2}{2} \cdot \frac{1}{12} - \cdots = 1 - \frac{\lambda}{2} + \frac{\lambda^2}{24} - \cdots$, and $\cos z = 1 - z^2/2 + z^4/24 - \cdots$ with $z^2 = \lambda$ agrees.

The kernel is continuous, symmetric and positive (Chapter 13), with eigenvalues $\mu_n = 1/((n - \frac{1}{2})\pi)^2$. The product formula gives $D(\lambda) = \prod_n (1 - \lambda/((n - \frac{1}{2})\pi)^2)$, and Euler's product $\cos z = \prod_n (1 - z^2/((n - \frac{1}{2})\pi)^2)$ with $z = \sqrt{\lambda}$ identifies it as $\cos \sqrt{\lambda}$. The zeros $((n - \frac{1}{2})\pi)^2$ are the poles found in Exercise 14.7.

Exercise 14.9. The exact value is $\cos 1 = 0.540302$. The quadrature determinants are $D_4 = 0.537406$, $D_8 = 0.539501$, $D_{16} = 0.540090$, $D_{32} = 0.540248$ and $D_{64} = 0.540288$, with errors 2.9×10^{-3} , 8.0×10^{-4} , 2.1×10^{-4} , 5.5×10^{-5} and 1.4×10^{-5} . Successive ratios are close to 3.6 to 3.9, approaching 4, so the error is about Cm^{-2} . The cause is the same as for the Green kernel in Table 14.1: $\min(x, t)$ has a corner along the diagonal, so the Gauss–Legendre rule loses its exponential accuracy. Splitting each integral at the diagonal, or using a rule adapted to the corner, restores fast convergence.

Exercise 14.10. *Jacobi's formula.* For an invertible matrix function $M(\lambda)$, $\frac{d}{d\lambda} \det M = \det M \operatorname{tr}(M^{-1}M')$. With $M = I - \lambda A$ and $M' = -A$,

$$\frac{D'(\lambda)}{D(\lambda)} = -\operatorname{tr}((I - \lambda A)^{-1}A).$$

The trace of the resolvent. By (14.12),

$$\begin{aligned} \int_a^b R(s, s; \lambda) ds &= \sum_{i,j} [(I - \lambda A)^{-1}]_{ij} \int_a^b b_j(s) a_i(s) ds \\ &= \sum_{i,j} [(I - \lambda A)^{-1}]_{ij} A_{ji} = \operatorname{tr}((I - \lambda A)^{-1}A). \end{aligned}$$

Combining the two gives the formula.

Check on B2. $D(\lambda) = 1 - \lambda/3$, so $D'/D = -\frac{1}{3}/(1 - \lambda/3) = -1/(3 - \lambda)$. And $\int_0^1 R(s, s; \lambda) ds = \int_0^1 \frac{3s^2}{3-\lambda} ds = \frac{1}{3-\lambda}$. The formula holds. The same identity holds for continuous kernels, with D and R from Theorem 14.7; it says that the logarithmic derivative of the determinant is minus the trace of the resolvent.

Exercise 14.11. Let $\lambda = k^2$ with k near π . Then $\sin k = -\sin(k - \pi) \approx -(k - \pi)$ and $\lambda - \pi^2 = (k - \pi)(k + \pi) \approx 2\pi(k - \pi)$, so

$$k \sin k \approx -\pi(k - \pi) \approx -\frac{\lambda - \pi^2}{2}.$$

The numerator of (14.11) tends to $\sin(\pi x_{<})\sin(\pi(1 - x_{>})) = \sin(\pi x_{<})\sin(\pi x_{>}) = \sin \pi x \sin \pi t$. Hence

$$R(x, t; \lambda) \approx -\frac{2 \sin(\pi x) \sin(\pi t)}{\lambda - \pi^2},$$

in agreement with Proposition 14.4: the residue is $-\varphi_1(x)\varphi_1(t)$ with $\varphi_1 = \sqrt{2} \sin \pi x$. For the solution, (14.2) gives

$$u(x) \approx f(x) - \frac{2\lambda}{\lambda - \pi^2} \sin(\pi x) \int_0^1 \sin(\pi t) f(t) dt + (\text{terms bounded near } \pi^2).$$

Unless the data is orthogonal to $\sin \pi t$, the solution is dominated by the first mode, with an amplitude that grows like $1/|\lambda - \pi^2|$. This is the near-resonance amplification measured by the operator norm in Chapter 13.

Exercise 14.12. Write $x - 3t = x \cdot 1 + 1 \cdot (-3t)$, so $a_1 = x$, $b_1 = 1$, $a_2 = 1$, $b_2 = -3t$. Then

$$A = \begin{pmatrix} \int_0^1 x & \int_0^1 1 \\ \int_0^1 (-3x)x & \int_0^1 (-3x) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 \\ -1 & -\frac{3}{2} \end{pmatrix},$$

$$D(\lambda) = \left(1 - \frac{\lambda}{2}\right)\left(1 + \frac{3\lambda}{2}\right) + \lambda^2 = \left(1 + \frac{\lambda}{2}\right)^2.$$

The adjugate of $I - \lambda A$ is $\begin{pmatrix} 1+3\lambda/2 & \lambda \\ -\lambda & 1-\lambda/2 \end{pmatrix}$, so

$$D(x, t; \lambda) = x\left(1 + \frac{3\lambda}{2}\right) - 3\lambda xt - \lambda - 3t\left(1 - \frac{\lambda}{2}\right) = (x - 3t) + \lambda\left(\frac{3}{2}(x + t) - 3xt - 1\right).$$

At $\lambda = -2$ the numerator is $x - 3t - 3(x + t) + 6xt + 2 = 2 - 2x - 6t + 6xt = 2(1 - x)(1 - 3t)$, which is not identically zero, so R has a pole of order exactly 2. The factor $1 - x$ is the single eigenfunction found in Chapter 13, and $1 - 3t$ is the eigenfunction of the adjoint kernel $t - 3x$.

For a Hermitian kernel, Proposition 14.4 writes R as a sum of terms $\varphi_n(x)\overline{\varphi_n(t)}/(\lambda_n - \lambda)$, each with a simple pole. A double pole needs a Jordan block, which a self-adjoint operator never has.

Chapter 15 Abel Equations and Weakly Singular Kernels

Exercise 15.1. Here $\alpha = \frac{1}{2}$, $f(0) = 0$ and $f'(t) = 2t$. By (15.5),

$$u(x) = \frac{1}{\pi} \int_0^x \frac{2t}{\sqrt{x-t}} dt = \frac{2}{\pi} \cdot \frac{\Gamma(2)\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2})} x^{3/2} = \frac{2}{\pi} \cdot \frac{4}{3} x^{3/2} = \frac{8}{3\pi} x^{3/2},$$

since $\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$. Check: by (15.6) with $\gamma = \frac{3}{2}$, $\int_0^x t^{3/2}(x-t)^{-1/2} dt = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)} x^2 = \frac{3\pi}{8} x^2$, and $\frac{8}{3\pi} \cdot \frac{3\pi}{8} x^2 = x^2$.

Exercise 15.2. The Abel operator with $\alpha = \frac{1}{2}$ raises powers by $\frac{1}{2}$, so try $u = ct$. By (15.6) with $\gamma = 1$,

$$\int_0^x \frac{ct}{\sqrt{x-t}} dt = c \frac{\Gamma(2)\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2})} x^{3/2} = c \cdot \frac{\sqrt{\pi}}{\frac{3}{4}\sqrt{\pi}} x^{3/2} = \frac{4c}{3} x^{3/2}.$$

So $c = \frac{3}{4}$. Uniqueness in Theorem 15.2 guarantees that the guess is the solution.

Exercise 15.3. With $f = 1$ we have $f(0) = 1$ and $f' = 0$, so (15.5) gives $u(x) = \frac{\sin(\pi\alpha)}{\pi} x^{\alpha-1}$. Check: $\int_0^x t^{\alpha-1}(x-t)^{-\alpha} dt = B(\alpha, 1-\alpha) = \pi/\sin(\pi\alpha)$ by Lemma 15.1, so the right side is 1. Since $\alpha-1 < 0$, u is unbounded as $x \rightarrow 0^+$, so it is not continuous on $[0, b]$; since $\alpha-1 > -1$, $\int_0^b u = \frac{\sin(\pi\alpha)}{\pi\alpha} b^\alpha$ is finite. This is the general form of the tautochrone solution.

Exercise 15.4. The Leibniz rule for $\frac{d}{dx} \int_0^x h(x, t) dt$ gives $h(x, x) + \int_0^x \partial_x h(x, t) dt$, and it is valid when h and $\partial_x h$ are continuous on the closed triangle $0 \leq t \leq x$. Here $h(x, t) = u(t)(x-t)^{-1/2}$, so two things go wrong.

- The boundary term $h(x, x) = u(x) \cdot 0^{-1/2}$ is infinite whenever $u(x) \neq 0$. The reader silently dropped it.
- The remaining integral $\int_0^x u(t)(x-t)^{-3/2} dt$ diverges at $t = x$ whenever $u(x) \neq 0$, so the displayed equation compares 1 with $-\infty$.

The conclusion “ u is negative” is therefore unfounded. The correct route is (15.5): with $f(0) = 0$ and $f' = 1$, $u(x) = \frac{1}{\pi} \int_0^x (x-t)^{-1/2} dt = \frac{2}{\pi} \sqrt{x}$, which is positive, as Example 15.2 checks by substitution. The substitution $t = x-r$ is what makes differentiation legitimate: after it, the singular factor $r^{-1/2}$ no longer depends on x .

Exercise 15.5. (a) From the example, $c = \sqrt{2g} T_0/\pi = \sqrt{19.62}/\pi \approx 4.4294/3.1416 \approx 1.4099 \text{ m}^{1/2}$. (b) The wire needs $u = c/\sqrt{y} \geq 1$, so $y \leq c^2 = 2gT_0^2/\pi^2 \approx 1.9879 \text{ m}$. (c) Arc length is $s(y) = \int_0^y c \eta^{-1/2} d\eta = 2c\sqrt{y}$, so at $y = c^2$ it is $2c^2 \approx 3.9758 \text{ m}$. At that height $ds/dy = 1$ and the wire is vertical, the top of a cycloid arch.

Exercise 15.6. Let u be continuous on $[0, b]$ with $|u| \leq M$. The double integral

$$\int_0^x \int_0^s (x-s)^{\beta-1} (s-t)^{\gamma-1} |u(t)| dt ds \leq M \int_0^x (x-s)^{\beta-1} \frac{s^\gamma}{\gamma} ds < \infty,$$

so the integrand is absolutely integrable over the triangle $0 < t < s < x$ and the order may be exchanged (Chapter 3). After exchanging, the inner integral is $\int_t^x (x-s)^{\beta-1} (s-t)^{\gamma-1} ds$. With $s = t + (x-t)r$ it becomes $(x-t)^{\beta+\gamma-1} \int_0^1 (1-r)^{\beta-1} r^{\gamma-1} dr = (x-t)^{\beta+\gamma-1} B(\gamma, \beta)$. Since $B(\gamma, \beta) = \Gamma(\beta)\Gamma(\gamma)/\Gamma(\beta+\gamma)$,

$$(I^\beta I^\gamma u)(x) = \frac{1}{\Gamma(\beta+\gamma)} \int_0^x (x-t)^{\beta+\gamma-1} u(t) dt = (I^{\beta+\gamma} u)(x).$$

For injectivity, suppose $I^\beta u = 0$. Pick an integer $m \geq \beta$. If $m > \beta$, apply $I^{m-\beta}$ to get $I^m u = I^{m-\beta} I^\beta u = 0$; if $m = \beta$ this already holds. By Cauchy's formula $I^m u$ is the m -fold repeated integral of u , and differentiating m times gives $u = 0$.

Exercise 15.7. For $n = 1$ the formula reads $K_1 = \Gamma(\frac{1}{2})(x-t)^{-1/2}/\Gamma(\frac{1}{2}) = (x-t)^{-1/2}$. Suppose it holds for n . For Volterra kernels $K_{n+1}(x, t) = \int_t^x K_1(x, s) K_n(s, t) ds$ (Chapter 7), so

$$K_{n+1}(x, t) = \frac{\Gamma(\frac{1}{2})^n}{\Gamma(\frac{n}{2})} \int_t^x (x-s)^{-1/2} (s-t)^{n/2-1} ds.$$

With $s = t + (x-t)r$ the integral is $(x-t)^{(n+1)/2-1} \int_0^1 (1-r)^{-1/2} r^{n/2-1} dr = (x-t)^{(n+1)/2-1} B(\frac{n}{2}, \frac{1}{2})$, and $B(\frac{n}{2}, \frac{1}{2}) = \Gamma(\frac{n}{2})\Gamma(\frac{1}{2})/\Gamma(\frac{n+1}{2})$. The factor $\Gamma(\frac{n}{2})$ cancels and

$$K_{n+1}(x, t) = \frac{\Gamma(\frac{1}{2})^{n+1}}{\Gamma(\frac{n+1}{2})} (x-t)^{(n+1)/2-1},$$

completing the induction. For $n \geq 2$ the exponent $n/2 - 1$ is nonnegative, so on $0 \leq t \leq x \leq b$

$$|\lambda^{n-1} K_n(x, t)| \leq \frac{|\lambda|^{n-1} \pi^{n/2} b^{n/2-1}}{\Gamma(\frac{n}{2})}.$$

The ratio of consecutive bounds for n and $n + 2$ is $|\lambda|^2 \pi b / (\frac{n}{2})$, which tends to 0, so the series of bounds converges and the Weierstrass M-test gives uniform convergence of $\sum_{n \geq 2} \lambda^{n-1} K_n$. The resolvent is therefore $R(x, t; \lambda) = (x - t)^{-1/2} + (\text{a continuous function})$ for every λ .

Exercise 15.8. By symmetry about the diagonal the integral is $2 \int_0^1 \int_0^x (x - t)^{-2\alpha} dt dx$. For $2\alpha < 1$ the inner integral is $x^{1-2\alpha}/(1-2\alpha)$, and then $2 \int_0^1 x^{1-2\alpha} dx / (1-2\alpha) = \frac{2}{(1-2\alpha)(2-2\alpha)} = \frac{1}{(1-2\alpha)(1-\alpha)}$. For $2\alpha \geq 1$ the inner integral $\int_0^x (x - t)^{-2\alpha} dt$ diverges for every $x > 0$, so the double integral is infinite. Hence $|x - t|^{-\alpha}$ is a Hilbert–Schmidt kernel on $[0, 1]$ if and only if $\alpha < \frac{1}{2}$. For $\frac{1}{2} \leq \alpha < 1$ the operator is still compact, but the proof must go through the Schur test and approximation, as in Theorem 15.6.

Exercise 15.9. For fixed $x \in [0, 1]$,

$$\int_0^1 \frac{dt}{\sqrt{|x - t|}} = \int_0^x \frac{dt}{\sqrt{x - t}} + \int_x^1 \frac{dt}{\sqrt{t - x}} = 2\sqrt{x} + 2\sqrt{1 - x}.$$

The function $\sqrt{x} + \sqrt{1 - x}$ has derivative $\frac{1}{2}(x^{-1/2} - (1 - x)^{-1/2})$, which vanishes only at $x = \frac{1}{2}$, where the value is $\sqrt{2}$; at the endpoints the value is 1. So $C_1 = 2\sqrt{2}$, and by symmetry of the kernel $C_2 = 2\sqrt{2}$. Lemma 15.5 gives $\|T\| \leq \sqrt{C_1 C_2} = 2\sqrt{2}$. The Neumann series converges when $|\lambda| \|T\| < 1$ (Chapter 6), which is guaranteed for $|\lambda| < 1/(2\sqrt{2}) \approx 0.354$. As always this is a sufficient condition.

Exercise 15.10. With $r = 1 - t$, $\int_0^1 e^t (1 - t)^{-1/2} dt = e \int_0^1 e^{-r} r^{-1/2} dr$, and with $r = s^2$ this is $2e \int_0^1 e^{-s^2} ds = e\sqrt{\pi} \operatorname{erf}(1) \approx 4.06015$. The code of the Computation box in Section 15.6 applies with $\cos$ replaced by $\exp$. The midpoint errors are about 0.41, 0.29, 0.21 and 0.15 for $n = 16, 32, 64, 128$, falling by $\sqrt{2}$ each time; product-integration errors are about 1.2×10^{-3} , 3.1×10^{-4} , 8.0×10^{-5} and 2.0×10^{-5} , falling by 4. The explanation is the one given for Table 15.1.

Exercise 15.11. Since $f(0) = 0$ and $f' = \cos$, (15.5) gives $u(x) = \frac{1}{\pi} \int_0^x \cos t (x - t)^{-1/2} dt$. Expand $\cos t = 1 - \frac{t^2}{2} + \mathcal{O}(t^4)$ and use (15.6): $\int_0^x (x - t)^{-1/2} dt = 2\sqrt{x}$ and $\int_0^x t^2 (x - t)^{-1/2} dt = \frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})} x^{5/2} = \frac{16}{15} x^{5/2}$. So

$$u(x) = \frac{1}{\pi} \left(2\sqrt{x} - \frac{1}{2} \cdot \frac{16}{15} x^{5/2} \right) + \mathcal{O}(x^{9/2}) = \frac{2}{\pi} \sqrt{x} - \frac{8}{15\pi} x^{5/2} + \mathcal{O}(x^{9/2}).$$

The leading term matches Example 15.2, as it should, because $\sin x \approx x$.

Exercise 15.12. Clearly $|f(x)| \leq \|u\|_\infty x^{1-\alpha}/(1-\alpha)$, so $f(0) = 0$. Take $0 \leq y < x \leq b$ and write

$$f(x) - f(y) = \int_0^y u(t)[(x-t)^{-\alpha} - (y-t)^{-\alpha}] dt + \int_y^x u(t)(x-t)^{-\alpha} dt.$$

The second integral is at most $\|u\|_\infty (x-y)^{1-\alpha}/(1-\alpha)$ in absolute value. In the first, $(y-t)^{-\alpha} \geq (x-t)^{-\alpha}$, so

$$\int_0^y [(y-t)^{-\alpha} - (x-t)^{-\alpha}] dt = \frac{y^{1-\alpha} - x^{1-\alpha} + (x-y)^{1-\alpha}}{1-\alpha} \leq \frac{(x-y)^{1-\alpha}}{1-\alpha},$$

because $y^{1-\alpha} \leq x^{1-\alpha}$. Multiplying by $\|u\|_\infty$ and adding the two bounds gives the claim. So data that come from a continuous u are Hölder continuous with exponent $1-\alpha$; data with a jump, or with a steeper start, cannot.

Exercise 15.13. The function e_ε increases from 0 to δ on $[0, \varepsilon]$ and is constant afterward, so $\max|e_\varepsilon| = \delta$. Because the equation is of Volterra type, the solution on $(0, \varepsilon]$ depends only on the data on $[0, \varepsilon]$. There

$$e_\varepsilon(x) = \frac{\delta}{2\sqrt{\varepsilon}} \cdot 2\sqrt{x},$$

and by Benchmark B4 the data $2\sqrt{x}$ come from the constant 1. So $w = \delta/(2\sqrt{\varepsilon})$ on $(0, \varepsilon]$, and uniqueness (Theorem 15.2, applied on $[0, \varepsilon]$ with the data $2\sqrt{x}$ times a constant) confirms that there is no other solution. As $\varepsilon \rightarrow 0$ the data perturbation stays of size δ while the solution perturbation $\delta/(2\sqrt{\varepsilon})$ grows without bound. No inequality of the form $\max|w| \leq C \max|e|$ can hold, so Abel inversion is unstable in the maximum norm. The instability is mild in the sense of Section 15.7: the growth is like $\varepsilon^{-1/2}$, half a power of the resolution, rather than the ε^{-1} that differentiation would give.

Chapter 16 Nonlinear Integral Equations

Exercise 16.1. (a) The integrand is $K(x, t)g(t, u(t))$ with $K(x, t) = x + t$ and $g(t, v) = v^3$, so the equation is Hammerstein or Fredholm type. (b) The integrand is $k(x, t, u(t))$ with $k(x, t, v) = \sin(xv)$, so the equation is of Urysohn type. It is not Hammerstein. Suppose $\sin(xv) = K(x, t)g(t, v)$ for all x and v at some fixed t . Putting $x = 1$ gives $g(t, v) = \sin v/K(1, t)$, so $\sin(xv)/\sin v = K(x, t)/K(1, t)$ would not depend on v . At $x = 2$ the ratio is $2 \cos v$, which does. (c) The unknown enters linearly, so this is a linear Volterra equation of the second kind with kernel e^{x-t} . (d) The factor $u(x)$ stands outside the integral, so the right side is not $f(x)$ plus an integral of $k(x, t, u(t))$. The equation is nonlinear but of neither form. Equations of this shape arise in radiative transfer.

Exercise 16.2. The right side does not depend on x , so every solution is a constant c , and $c = 1 + \lambda c^2$. The quadratic $\lambda c^2 - c + 1 = 0$ has discriminant $1 - 4\lambda$. For $\lambda < \frac{1}{4}$ (and $\lambda \neq 0$) it has the 2 real roots $c = (1 \pm \sqrt{1 - 4\lambda})/(2\lambda)$; for $\lambda < 0$ one is positive and one negative. For $\lambda = \frac{1}{4}$ the double root is $c = 2$. For $\lambda > \frac{1}{4}$ there is no real root and no solution. Check at $\lambda = \frac{1}{4}$: $1 + \frac{1}{4} \cdot 4 = 2$.

Exercise 16.3. $u_1 = 1 + \int_0^x 1 \, dt = 1 + x$. Next $u_2 = 1 + \int_0^x (1 + t)^2 \, dt = 1 + x + x^2 + \frac{1}{3}x^3$. For u_3 expand

$$(1 + t + t^2 + \frac{1}{3}t^3)^2 = 1 + 2t + 3t^2 + \frac{8}{3}t^3 + \frac{5}{3}t^4 + \frac{2}{3}t^5 + \frac{1}{9}t^6$$

and integrate: $u_3 = 1 + x + x^2 + x^3 + \frac{2}{3}x^4 + \frac{1}{3}x^5 + \frac{1}{9}x^6 + \frac{1}{63}x^7$. The solution is $1/(1 - x) = 1 + x + x^2 + x^3 + \dots$. So u_1 is right through x^1 , u_2 through x^2 and u_3 through x^3 . By induction u_n is right through x^n : if u_{n-1} agrees with $1/(1 - x)$ up to an error of order x^n , then u_{n-1}^2 agrees with $1/(1 - x)^2$ up to order x^n , and integrating raises that to order x^{n+1} . The iterates are not partial sums of the Taylor series: u_2 contains $\frac{1}{3}x^3$, which is neither 0 nor the correct coefficient 1.

Exercise 16.4. Here $K = e^{-|x-t|}$ and $g(t, v) = \sin v$, with $|g| \leq 1$ and Lipschitz constant $L = 1$ for all v . The row integral is $\int_0^1 e^{-|x-t|} \, dt = 2 - e^{-x} - e^{-(1-x)}$, which is largest at $x = \frac{1}{2}$, so $\kappa = 2(1 - e^{-1/2}) \approx 0.7869$. Take $M = 1$ and any $r \geq |\lambda|\kappa$; then the self-map

condition holds, and the contraction condition is $|\lambda|\kappa < 1$, that is, $|\lambda| < 1/0.7869 \approx 1.271$. Since r can be as large as we like, uniqueness holds among all continuous functions, not only in a ball.

Exercise 16.5. (a) With $A = 1$, on $|v - 1| \leq \beta$ and $0 \leq t \leq 1$ the integrand is at most $M = (1 + \beta)^2 + 1$. The guaranteed length is $h = \beta/((1 + \beta)^2 + 1)$ (it is below 1 for all β). Setting the derivative to zero gives $(1 + \beta)^2 + 1 - 2\beta(1 + \beta) = 0$, that is, $2 - \beta^2 = 0$, so $\beta = \sqrt{2}$ and

$$h = \frac{\sqrt{2}}{(1 + \sqrt{2})^2 + 1} = \frac{\sqrt{2}}{4 + 2\sqrt{2}} = \frac{\sqrt{2} - 1}{2} \approx 0.2071.$$

(b) Since $u' = u^2 + x^2 \geq u^2$ and $u(0) = 1$, comparison with $v' = v^2$, $v(0) = 1$ gives $u(x) \geq 1/(1 - x)$ as long as both exist, so u blows up at some $x^* \leq 1$. Integrating $w = 1/u$, which satisfies $w' = -1 - x^2 w^2$ and $w(0) = 1$, until w reaches 0 gives $x^* \approx 0.9698$. The local theorem guaranteed about a fifth of the true interval.

Exercise 16.6. For $x \leq s$ both sides are 0. For $x > s$, $\sqrt{u_s(t)} = 0$ on $[0, s]$ and $\frac{1}{2}(t - s)$ on $[s, x]$, so the integral is $\int_s^x \frac{1}{2}(t - s) dt = \frac{1}{4}(x - s)^2 = u_s(x)$. The function is continuous, with a continuous derivative, at $x = s$. Here $f = 0$ and $k(x, t, v) = \sqrt{v}$, so any tube $|v - f(t)| \leq \beta$ contains $v = 0$, and $|\sqrt{v} - \sqrt{w}| \leq L|v - w|$ fails for $w = 0$ and small $v > 0$ because $\sqrt{v}/v = 1/\sqrt{v}$ is unbounded. The theorem does not apply, and uniqueness indeed fails.

Exercise 16.7. The corollary requires $|k(x, t, v) - k(x, t, w)| \leq L|v - w|$ with a single L for all real v and w . For $k = v^2$ we have $|v^2 - w^2| = |v + w||v - w|$, and no single constant bounds $|v + w|$. Being Lipschitz on every bounded set is a weaker property: the constant depends on the set, and the solution may leave any given set. What is true is Theorem 16.4: with $a = \lambda = 1$, Example 16.3 gives existence on $[0, \frac{1}{4}]$, and continuation extends the solution $1/(1 - x)$ to $[0, 1)$ but not further. No continuous solution exists on $[0, 2]$, since any such solution would agree with $1/(1 - x)$ on $[0, 1)$ and so be unbounded near $x = 1$. The Picard iterates are polynomials and exist on $[0, 2]$, but Section 16.5 shows that they increase without bound for every $x \geq 1$, so they do not compute anything there.

Exercise 16.8. The smaller root of $c^2 - 8c + 8 = 0$ is $c^* = 4 - 2\sqrt{2} \approx 1.1715729$; it is the one near $c_0 = 1$. With $F(c) = c - 1 - c^2/8$ and $F'(c) = 1 - c/4$, Newton's method gives:

- $F(1) = -\frac{1}{8}$, $F'(1) = \frac{3}{4}$, so $c_1 = 1 + \frac{1/8}{3/4} = \frac{7}{6} \approx 1.1666667$.
- $F(\frac{7}{6}) = \frac{1}{6} - \frac{49}{288} = -\frac{1}{288}$, $F'(\frac{7}{6}) = \frac{17}{24}$, so $c_2 = \frac{7}{6} + \frac{24}{288 \cdot 17} = \frac{7}{6} + \frac{1}{204} = \frac{239}{204} \approx 1.1715686$.

The errors are about 4.9×10^{-3} and 4.2×10^{-6} ; the second is roughly the square of the first times 0.18, the quadratic pattern. Picard iteration $c_{k+1} = 1 + c_k^2/8$ gives $c_1 = 1.125$ and $c_2 = 1 + 1.265625/8 \approx 1.158203$, with errors about 0.047 and 0.013, shrinking by the factor $|\partial_c(1 + c^2/8)| = c^*/4 \approx 0.29$ per step.

Exercise 16.9. Since $u^* = \Phi(u^*)$ and $u_{n+1} = \Phi(u_n)$, we have $\|u^* - u_{n+1}\| \leq q\|u^* - u_n\|$ and $\|u_{n+1} - u_n\| \leq q\|u_n - u_{n-1}\|$. The triangle inequality gives

$$\|u^* - u_n\| \leq \|u^* - u_{n+1}\| + \|u_{n+1} - u_n\| \leq q\|u^* - u_n\| + q\|u_n - u_{n-1}\|,$$

and since $1 - q > 0$, $\|u^* - u_n\| \leq \frac{q}{1-q}\|u_n - u_{n-1}\|$. The a priori bound needs q^n and the first step, and it usually overestimates badly because q is a worst-case constant. The a posteriori bound uses the most recent step, which the computation has in hand, so it gives a sharper and directly usable stopping rule: stop when $\frac{q}{1-q}\|u_n - u_{n-1}\|$ falls below the tolerance.

Exercise 16.10. Every solution has the form $u = 1 + \lambda cx$ with $c = \int_0^1 tu(t)^2 dt$, and conversely such a u is a solution exactly when c satisfies

$$c = \int_0^1 t(1 + \lambda ct)^2 dt = \frac{1}{2} + \frac{2\lambda c}{3} + \frac{\lambda^2 c^2}{4},$$

that is, $\frac{\lambda^2}{4}c^2 + (\frac{2\lambda}{3} - 1)c + \frac{1}{2} = 0$. Distinct roots c give distinct solutions, so the number of solutions equals the number of real roots. The discriminant is

$$\Delta(\lambda) = \left(\frac{2\lambda}{3} - 1\right)^2 - \frac{\lambda^2}{2} = \left(\frac{2\lambda}{3} - 1 - \frac{\lambda}{\sqrt{2}}\right)\left(\frac{2\lambda}{3} - 1 + \frac{\lambda}{\sqrt{2}}\right).$$

For $\lambda > 0$ the first factor is negative (its λ -coefficient $\frac{2}{3} - \frac{1}{\sqrt{2}}$ is negative), and the second factor is negative for $\lambda < \lambda_*$ and positive for $\lambda > \lambda_*$, where $\lambda_* = 1/(\frac{2}{3} + \frac{1}{\sqrt{2}}) = 3\sqrt{2}/(3 + 2\sqrt{2}) = 3\sqrt{2}(3 - 2\sqrt{2}) = 9\sqrt{2} - 12 \approx 0.7279$. So $\Delta > 0$ for $0 < \lambda < \lambda_*$ (2 roots), $\Delta = 0$ at λ_* (1 root) and $\Delta < 0$ beyond (0 roots). Example 16.2 is the case $\lambda = \frac{1}{4}$.

Exercise 16.11. The equation $\theta = \sqrt{6} \cosh(\theta/4)$ has roots $\theta \approx 3.37351$ and $\theta \approx 6.57657$, found by bisection on either side of $\theta_c \approx 4.7987$. The exact maxima $u(\frac{1}{2}) = 2 \log \cosh(\theta/4)$ are about 0.64015 and 1.97527. Newton from $A = 0$ converges to the lower solution with computed maximum about 0.64026, and from $A = 4$ to the upper one with maximum about 1.97482. The differences, about 10^{-4} and 4×10^{-4} , are discretization errors; they fall by about 4 when N is doubled. The upper solution has the larger discretization error because it is more sharply curved.

Exercise 16.12. The integrand $k(x, t, v) = \sin v$ satisfies a Lipschitz condition with $L = 1$ for all real v , so Corollary 16.5 gives a unique continuous solution on every $[0, A]$, and Picard iteration converges there. Differentiating gives $u' = 1 + \sin u$ with $u(0) = 0$. Since $1 + \sin u = 2 \cos^2(\frac{u}{2} - \frac{\pi}{4})$, separating variables gives $\tan(\frac{u}{2} - \frac{\pi}{4}) = x + C$, and $u(0) = 0$ gives $C = -1$. So $u(x) = \frac{\pi}{2} + 2 \arctan(x - 1)$. Check: $u(0) = \frac{\pi}{2} - \frac{\pi}{2} = 0$, $u' = 2/(1 + (x - 1)^2)$, and $1 + \sin u = 1 + \cos(2 \arctan(x - 1)) = 1 + \frac{1 - (x-1)^2}{1 + (x-1)^2} = \frac{2}{1 + (x-1)^2}$. The solution increases toward $\frac{3\pi}{2}$, where $1 + \sin u = 0$, and never reaches it, so it is bounded, as the global theorem allows.

Exercise 16.13. If u is a continuous solution, define $c_i = \int_a^b b_i(t)g(t, u(t)) dt$. The equation says $u = f + \lambda \sum_j a_j c_j$, and substituting this expression for u into the definition of c_i shows that $\mathbf{c}$ solves the algebraic system. Conversely, if $\mathbf{c}$ solves the system, put $u = f + \lambda \sum_j a_j c_j$, which is continuous. The system says exactly that $\int_a^b b_i g(t, u) dt = c_i$, so the right side of the integral equation equals $f + \lambda \sum_i a_i c_i = u$, and u is a solution. Starting from u , forming $\mathbf{c}$ and rebuilding gives back $f + \lambda \sum a_j c_j = u$. Starting from $\mathbf{c}$, building u and forming $\int b_i g(t, u)$ gives back $\mathbf{c}$ by the system. So the correspondence is one-to-one, provided the a_i are linearly independent; if they are not, distinct $\mathbf{c}$ can give the same u , and the statement should be read with $\mathbf{c}$ replaced by the function $\sum a_j c_j$. Example 16.2 is the case $m = 1$, where the system is one quadratic equation.

Chapter 17 Cauchy Principal Values and Singular Equations

Exercise 17.1. With $u(t) = t$ the regular part is $\int_{-1}^1 (t-x)/(t-x) dt = \int_{-1}^1 1 dt = 2$. By the subtraction formula (17.4),

$$\text{p.v.} \int_{-1}^1 \frac{t}{t-x} dt = 2 + x \log \frac{1-x}{1+x}.$$

At $x = 0$ the integrand is identically 1 (the singularity is removable) and the integral is 2, which the formula reproduces because the logarithm term carries the factor x . At $x = 0.3$ the value is $2 + 0.3 \log(0.7/1.3) \approx 1.8143$.

Exercise 17.2. For $0 < \varepsilon < 1$,

$$\int_0^{1-\varepsilon} \frac{dt}{t-1} + \int_{1+\varepsilon}^3 \frac{dt}{t-1} = (\log \varepsilon - \log 1) + (\log 2 - \log \varepsilon) = \log 2.$$

The expression does not depend on ε , so its limit is $\log 2$. Formula (17.3) with $a = 0$, $b = 3$, $x = 1$ gives $\log((3-1)/(1-0)) = \log 2$ as well.

Exercise 17.3. Oddness about $t = x$ means $g(x+d) = -g(x-d)$. It gives cancellation only when every point $x+d$ in the interval is matched by the point $x-d$, that is, only on an interval of the form $[x-L, x+L]$. The interval $[-1, 1]$ has this form only when $x = 0$.

For $x > 0$, split $[-1, 1]$ into the symmetric part $[2x-1, 1]$ and the unmatched part $[-1, 2x-1]$. With symmetric cuts the symmetric part contributes 0, and the unmatched part contains no singularity, so

$$\text{p.v.} \int_{-1}^1 \frac{dt}{t-x} = \int_{-1}^{2x-1} \frac{dt}{t-x} = \log(1-x) - \log(1+x).$$

For $x < 0$ the unmatched part is $[2x+1, 1]$ and the same computation gives the same formula. The value is negative for $x > 0$ and positive for $x < 0$, and it is zero only at $x = 0$. A quick numerical test at $x = 0.3$ gives $\log(0.7/1.3) \approx -0.619$, which is far from zero.

Exercise 17.4. (a) An antiderivative of $(t-x)^{-2}$ is $-1/(t-x)$. So

$$\int_{-1}^{x-\varepsilon} \frac{dt}{(t-x)^2} = \frac{1}{\varepsilon} - \frac{1}{1+x}, \quad \int_{x+\varepsilon}^1 \frac{dt}{(t-x)^2} = -\frac{1}{1-x} + \frac{1}{\varepsilon}.$$

The sum is $2/\varepsilon - (1/(1+x) + 1/(1-x)) = 2/\varepsilon - 2/(1-x^2)$, which tends to $+\infty$ as $\varepsilon \rightarrow 0^+$. So the principal value does not exist. (The number $-2/(1-x^2)$ left after discarding $2/\varepsilon$ is the Hadamard finite part, which is a different definition.)

(b) With symmetric cuts,

$$\int_{-1}^{x-\varepsilon} \frac{dt}{x-t} + \int_{x+\varepsilon}^1 \frac{dt}{t-x} = (\log(1+x) - \log \varepsilon) + (\log(1-x) - \log \varepsilon) = \log \frac{1-x^2}{\varepsilon^2},$$

which tends to $+\infty$.

(c) In both cases the integrand is even about $t = x$ and positive, so the contributions from the two sides add instead of cancelling. The principal value of $\int dt/(t-x)$ exists only because the integrand changes sign across x in a symmetric way.

Exercise 17.5. (a) The logarithm is unbounded near $t = x$ but integrable, since $\int_0^\varepsilon |\log s| ds = \varepsilon(1 - \log \varepsilon) \rightarrow 0$. Both one-sided limits exist, so this is a convergent improper integral, type (ii).

(b) $\int_0^\varepsilon s^{-1/2} ds = 2\sqrt{\varepsilon} \rightarrow 0$, so both one-sided limits exist: type (ii).

(c) $\sin(s)/s \rightarrow 1$ as $s \rightarrow 0$, so the singularity is removable and the integrand extends to a continuous function: type (i).

(d) Near $t = x$ the integrand behaves like $e^x/(t-x)$, which is not integrable on either side, so the improper integral fails. By Theorem 17.3 the principal value exists and equals $e^x(\text{Ei}(1-x) - \text{Ei}(-1-x))$: type (iii).

(e) By Exercise 17.4 even symmetric cuts give $+\infty$: type (iv). It has a Hadamard finite part, but that is a further convention not covered in this chapter.

Exercise 17.6. For $t \neq x$ the mean value theorem gives a point ξ_t strictly between x and t with $u(t) - u(x) = u'(\xi_t)(t-x)$, so $q(t) = u'(\xi_t)$. As $t \rightarrow x$ we have $\xi_t \rightarrow x$, and continuity of u' gives $q(t) \rightarrow u'(x)$. Defining $q(x) = u'(x)$ therefore makes q continuous on $[-1, 1]$.

The subtracted rule is the quadrature rule applied to q plus an exact logarithm. A quadrature rule samples the integrand at its nodes, so if $x = t_k$ the correct sample is $q(t_k) = u'(x)$ and the k th term is $w_k u'(x)$. Omitting the term would change the result by $w_k u'(x)$, which is an error of the same size as the weights, so the rule would no longer converge at its usual rate. If u' is not available in closed form, a difference quotient of u at x is a reasonable substitute with its own small error, and the cleanest fix is to choose a rule whose nodes avoid x .

Exercise 17.7. The reference value from (17.6) is

$$e^{0.5}(\text{Ei}(0.5) - \text{Ei}(-1.5)) = 0.9137864317 \dots$$

The subtracted rule with 8 nodes,

$$\sum_{j=1}^8 w_j \frac{e^{t_j} - e^{0.5}}{t_j - 0.5} + e^{0.5} \log \frac{0.5}{1.5},$$

agrees with it to about 13 digits, because the regular part is an entire function of t and Gauss–Legendre quadrature converges faster than any power of $1/N$ for such integrands.

The naive rule samples $e^t/(t - 0.5)$, whose size near $t = 0.5$ is limited only by the distance from 0.5 to the nearest node. That distance varies irregularly with N , so the naive values vary irregularly too, with errors of order 1 or larger, as in Table 17.1 for $x = 0.3$. The naive rule has no reason to converge, because its error analysis assumes a smooth integrand.

Exercise 17.8. Put $\varepsilon_1 = x - \phi(s_0 - \varepsilon)$ and $\varepsilon_2 = \phi(s_0 + \varepsilon) - x$. Since ϕ' is Lipschitz, Taylor's theorem gives $\varepsilon_1 = \phi'(s_0)\varepsilon + \mathcal{O}(\varepsilon^2)$ and $\varepsilon_2 = \phi'(s_0)\varepsilon + \mathcal{O}(\varepsilon^2)$, so $\varepsilon_1/\varepsilon_2 \rightarrow 1$. The substitution $t = \phi(s)$ is an ordinary change of variables on each of the two pieces that remain after the cut, so

$$\left(\int_c^{s_0-\varepsilon} + \int_{s_0+\varepsilon}^d \right) \frac{u(\phi(s))\phi'(s)}{\phi(s) - x} ds = \left(\int_a^{x-\varepsilon_1} + \int_{x+\varepsilon_2}^b \right) \frac{u(t)}{t - x} dt.$$

Now subtract $u(x)$ as in Theorem 17.3. The regular part is absolutely integrable, so its integral over $[a, x - \varepsilon_1] \cup [x + \varepsilon_2, b]$ tends to its integral over $[a, b]$ no matter how $\varepsilon_1, \varepsilon_2 \rightarrow 0$. The singular part is $u(x)$ times

$$\left(\int_a^{x-\varepsilon_1} + \int_{x+\varepsilon_2}^b \right) \frac{dt}{t - x} = \log \frac{\varepsilon_1}{\varepsilon_2} + \log \frac{b - x}{x - a},$$

by the calculation behind (17.2). Since $\log(\varepsilon_1/\varepsilon_2) \rightarrow 0$, the right-hand side tends to p.v. $\int_a^b u(t)/(t - x) dt$. Hence the principal value in s exists and equals the principal value in t . The key point is that the asymmetry of the cut is of second order in ε , so it perturbs a logarithm of a ratio that tends to 1.

Exercise 17.9. The recurrence in the proof with $n = 1$ reads $I_2 + I_0 = 2 \cos \varphi I_1$. With $I_0 = 0$ and $I_1 = 1$ this gives $I_2 = 2 \cos \varphi = \sin 2\varphi / \sin \varphi$, in agreement with (17.9). Under the substitution $t = \cos \theta$, $x = \cos \varphi$ of (17.8), the numerator $\cos 2\theta$ is $T_2(t) = 2t^2 - 1$, so

$$\frac{1}{\pi} \text{p.v.} \int_{-1}^1 \frac{2t^2 - 1}{(t - x)\sqrt{1 - t^2}} dt = 2x = U_1(x),$$

which is (17.11) with $n = 2$.

Exercise 17.10. First expand the right side in Chebyshev polynomials of the second kind. Since $U_2(x) = 4x^2 - 1$ and $U_0 = 1$,

$$4x^2 = U_2(x) + U_0(x).$$

By (17.14) the general solution is

$$u(x) = \frac{T_3(x) + T_1(x) + C}{\sqrt{1-x^2}} = \frac{(4x^3 - 3x) + x + C}{\sqrt{1-x^2}} = \frac{4x^3 - 2x + C}{\sqrt{1-x^2}}, \quad C \in \mathbb{R}.$$

For boundedness at $x = 1$ the numerator must vanish there: $4 - 2 + C = 0$, so $C = -2$. Then $4x^3 - 2x - 2 = (x-1)(4x^2 + 4x + 2)$, and

$$u(x) = \frac{(x-1)(4x^2 + 4x + 2)}{\sqrt{1-x^2}} = -(4x^2 + 4x + 2)\sqrt{\frac{1-x}{1+x}}.$$

Check: by Theorem 17.6,

$$H[T_3/\sqrt{1-t^2}] = U_2, \quad H[T_1/\sqrt{1-t^2}] = U_0, \quad H[1/\sqrt{1-t^2}] = 0,$$

so $Hu = U_2 + U_0 = 4x^2$ for every C . The numerator at $x = -1$ is $-4 + 2 - 2 = -4 \neq 0$, so this solution is unbounded at the leading edge, as expected: the condition (17.16) fails because $\int_{-1}^1 4t^2/\sqrt{1-t^2} dt = 2\pi \neq 0$.

Exercise 17.11. From Example 17.5 the general solution is $u = (\frac{1}{2}T_2(x) + C)/\sqrt{1-x^2}$. With $t = \cos \theta$,

$$\int_{-1}^1 u(t) dt = \int_0^\pi \left(\frac{1}{2} \cos 2\theta + C\right) d\theta = \pi C,$$

so the circulation condition gives $C = \Gamma/\pi$ and $u = (2x^2 - 1 + 2\Gamma/\pi)/(2\sqrt{1-x^2})$. By Example 17.5 the solution is bounded at both ends exactly when $C = -\frac{1}{2}$, that is $\Gamma = -\pi/2$.

Consistency check: $\int_{-1}^1 -\sqrt{1-t^2} dt = -\pi/2$, the negative of the area of a half disk of radius 1.

Exercise 17.12. Every solution has the form $u = P/\sqrt{1-x^2}$ with

$$P(x) = \sum_{k=0}^m b_k T_{k+1}(x) + C.$$

Since P is a polynomial, u is bounded near $x = 1$ exactly when $P(1) = 0$, because otherwise $|u| \sim |P(1)|/\sqrt{2(1-x)} \rightarrow \infty$; and if $P(1) = 0$ then $P(x) = (1-x)R(x)$ and $u = R(x)\sqrt{1-x}/\sqrt{1+x}$ is bounded near 1. The same holds at $x = -1$. So a bounded solution exists iff some C makes $P(1) = P(-1) = 0$.

Using $T_{k+1}(1) = 1$ and $T_{k+1}(-1) = (-1)^{k+1}$,

$$P(1) = \sum_k b_k + C, \quad P(-1) = \sum_k (-1)^{k+1} b_k + C.$$

Both vanish iff $C = -\sum_k b_k$ and $\sum_k b_k = \sum_k (-1)^{k+1} b_k$, that is, iff $\sum_k (1 + (-1)^k) b_k = 0$, which says $\sum_{k \text{ even}} b_k = 0$.

Now compute the weighted integral of f . With $t = \cos \theta$,

$$\int_{-1}^1 \frac{U_k(t)}{\sqrt{1-t^2}} dt = \int_0^\pi \frac{\sin(k+1)\theta}{\sin \theta} d\theta,$$

which equals π for even k and 0 for odd k . (Write $\sin(k+1)\theta/\sin \theta$ as a sum of cosines: for even $k = 2j$ it is $1 + 2 \sum_{i=1}^j \cos 2i\theta$, and for odd $k = 2j+1$ it is $2 \sum_{i=0}^j \cos(2i+1)\theta$.) Hence $\int_{-1}^1 f(t)/\sqrt{1-t^2} dt = \pi \sum_{k \text{ even}} b_k$, and the condition above is exactly (17.16). When it holds, C is forced to equal $-\sum_k b_k$, so the bounded solution is unique.

Chapter 18 Quadrature and the Nyström Method

Exercise 18.1. By (18.9) the Nyström solution of $u = x + \lambda \int_0^1 xt u$ is $u_N(x) = x/(1 - \lambda Q[t^2])$ for any rule Q . The midpoint rule gives $Q[t^2] = \frac{1}{2}(\frac{1}{16} + \frac{9}{16}) = \frac{5}{16}$. With $\lambda = \frac{1}{2}$,

$$u_N(x) = \frac{x}{1 - \frac{5}{32}} = \frac{32}{27} x.$$

The exact solution is $u(x) = 3x/(3 - \lambda) = \frac{6}{5}x$. At $x = 1$ the error is $u(1) - u_N(1) = \frac{6}{5} - \frac{32}{27} = \frac{162-160}{135} = \frac{2}{135} \approx 0.0148$. The composite midpoint rule has error $\int_0^1 g - Qg = \frac{h^2}{24}g''(\xi)$ on $[0, 1]$, which for $g = t^2$ and $h = \frac{1}{2}$ is exactly $\frac{1}{48}$. So the rule underestimates $\frac{1}{3}$ by $\frac{1}{48}$, the discrete characteristic value $1/Q[t^2] = 3.2$ lies above the true value 3, and u_N is slightly too small.

Exercise 18.2. The error formula for the composite trapezoid rule, $\int_0^1 g - Q_m^T g = -\frac{h^2}{12}g''(\xi)$, is exact for $g(t) = t^2$ because $g'' \equiv 2$. Hence

$$Q_m^T[t^2] = \frac{1}{3} + \frac{2h^2}{12} = \frac{1}{3} + \frac{h^2}{6}.$$

By (18.9), $u_N(x) = x/(1 - \lambda Q[t^2])$, and by Lemma 18.1 the Nyström system is singular exactly when $1 - \lambda Q[t^2] = 0$, that is at

$$\lambda^{(m)} = \frac{1}{\frac{1}{3} + \frac{h^2}{6}} = \frac{6}{2 + h^2} = 3\left(1 + \frac{h^2}{2}\right)^{-1} = 3 - \frac{3}{2}h^2 + \frac{3}{4}h^4 - \dots.$$

For $m = 2$, $h = \frac{1}{2}$ and $\lambda^{(2)} = 6/2.25 = \frac{8}{3}$, matching Example 18.1. The discrete characteristic value approaches the true value 3 from below at the rate h^2 , the order of the rule.

Exercise 18.3. Let $\mu \neq 0$ and $\lambda = 1/\mu$. The equation $T_N \varphi = \mu \varphi$ is the same as $\varphi = \lambda T_N \varphi$, and $\mathbf{A} \mathbf{v} = \mu \mathbf{v}$ is the same as $(I - \lambda \mathbf{A}) \mathbf{v} = 0$. Lemma 18.1 with $f = 0$ shows that a nonzero continuous solution of $\varphi = \lambda T_N \varphi$ has nonzero node values $\mathbf{v} = (\varphi(t_1), \dots, \varphi(t_N))^T$ solving

$(I - \lambda \mathbf{A})\mathbf{v} = 0$; if all node values vanished, the right side $\lambda T_N \varphi$ would vanish identically and so would φ . Conversely a nonzero $\mathbf{v}$ with $\mathbf{A}\mathbf{v} = \mu \mathbf{v}$ gives

$$\varphi(x) = \lambda \sum_{j=1}^N w_j K(x, t_j) v_j = \frac{1}{\mu} \sum_{j=1}^N w_j K(x, t_j) v_j,$$

which is continuous, has $\varphi(t_i) = \mu^{-1}(\mathbf{A}\mathbf{v})_i = v_i$, and satisfies $T_N \varphi = \mu \varphi$. So the nonzero eigenvalues coincide, and the eigenfunctions of T_N are exactly the Nyström interpolants of the eigenvectors of $\mathbf{A}$. The eigenvalue $\mu = 0$ is excluded because T_N has an infinite-dimensional null space in $C[a, b]$ (every continuous function vanishing at the nodes), while $\mathbf{A}$ may be invertible.

Exercise 18.4. The quantity $u_i - f(t_i) - \lambda \sum_j A_{ij} u_j$ is the residual of the linear system (18.5). Any backward-stable solver makes it of the order of rounding error times $\|\mathbf{A}\| \|\mathbf{u}\|$, whatever the quadrature error is. The error of the method, on the other hand, is governed by the error equation (18.11), whose right side is the quadrature error $(T - T_N)u$; that term never enters the node residual. So the report has verified only that the 65×65 system (there are 65 nodes for 64 Simpson panels) was solved correctly.

To assess the solution, evaluate the Nyström interpolant at points that are not nodes, compute $r_N(x) = u_N(x) - f(x) - \lambda \int K(x, t) u_N(t) dt$ there with an integration rule much more accurate than Simpson's (for a smooth kernel, a high-order Gauss–Legendre rule; for a kink, split the integral at the kink), and relate it to the error through (18.18). Independently, repeat the computation with 128 and 256 panels and check that the differences $u_{64} - u_{128}$ and $u_{128} - u_{256}$ decrease at the observed order 4. If the kernel has a kink on the diagonal, expect order 2 instead (Section 18.6).

Exercise 18.5. (a) For fixed x the integrand $t \mapsto e^{-(x-t)^2} u(t)$ is analytic whenever u is, and $u = f + \lambda T u$ inherits smoothness from f and the kernel. Gauss–Legendre then converges geometrically, as in Figure 18.1, and a dozen nodes will usually reach rounding level.

(b) The integrand $t \mapsto |x - t|u(t)$ has a jump in its first derivative at $t = x$, exactly like the Green's function kernel of Section 18.6. Every rule then shows order 2 in the Nyström formula, so the cheap trapezoid rule is as good as Simpson or Gauss. To do better, subtract the corner: $\int |x - t|u(t) dt = \int |x - t|(u(t) - u(x)) dt + u(x) \int_0^1 |x - t| dt$ with $\int_0^1 |x - t| dt = \frac{1}{2}(x^2 + (1 - x)^2)$; or split the integral at $t = x$ and use Gauss–Legendre on each piece (this requires a different set of nodes for each x , so it fits product integration better than plain Nyström).

(c) The kernel is unbounded at $t = x$, so $K(t_i, t_i)$ is undefined and the Nyström matrix cannot even be formed. The singularity is integrable, and product integration (integrate the logarithm exactly against a piecewise polynomial approximation of u) or subtraction of $u(x) \int \log|x - t| dt$ is required.

(d) Over a full period the trapezoid rule is exact for trigonometric polynomials of degree less than the number of nodes, and for analytic periodic integrands its error decreases geometrically. Since K and f are analytic and periodic, so is the integrand, and the equally spaced trapezoid rule is the right choice. Chapter 23 uses this fact for boundary integral equations on smooth closed curves.

Exercise 18.6. The largest errors on the fine grid with 4 Gauss–Legendre nodes are about 7.97×10^{-9} for $\lambda = 0.5$, 2.61×10^{-8} for $\lambda = 0.7$ and 8.86×10^{-8} for $\lambda = 0.73$. Increasing N drives each of them down geometrically, so the rate is unchanged.

The error equation (18.11) explains the trend. The quadrature error $(T - T_N)u$ does not depend on λ , because $u = \cos x$ is the same in all three runs. What changes is the factor $|\lambda|(I - \lambda T_N)^{-1}$. The largest operator eigenvalue of the kernel e^{xt} is $\mu_1 \approx 1.353$, so $I - \lambda T$ has the eigenvalue $1 - \lambda\mu_1$, which equals 0.32, 0.053 and 0.012 for the three values of λ . The component of the consistency error along the first eigenfunction is amplified by the reciprocal of that number. The observed errors grow by factors of about 3.3 and 3.4, less than the factors 6 and 4.3 by which $1/(1 - \lambda\mu_1)$ grows, because only part of the consistency error lies along the first eigenfunction. The bound (18.12) is honest but not tight.

Exercise 18.7. If $u_m(1) = u(1) + Ch^p + \mathcal{O}(h^{p+1})$ with $h = 1/m$, then

$$\frac{u_{16}(1) - u_8(1)}{u_{32}(1) - u_{16}(1)} = \frac{C(16^{-p} - 8^{-p})}{C(32^{-p} - 16^{-p})} + \dots = 2^p + \dots.$$

The differences are $u_{16} - u_8 = 0.0028264$ and $u_{32} - u_{16} = 0.0007040$, with ratio 4.015, so $p \approx \log_2 4.015 \approx 2.01$, as expected for the trapezoid rule on a smooth integrand. Knowing $p = 2$, the combination $\frac{1}{3}(4u_{32} - u_{16})$ cancels the h^2 term:

$$\frac{1}{3}(4 \times 0.5400679 - 0.5393639) = \frac{1}{3} \times 1.6209077 = 0.5403026,$$

which agrees with $\cos 1 = 0.5403023$ to about 3×10^{-7} (the rounding of the given data limits the last digit). The unextrapolated value $u_{32}(1)$ has error 2.3×10^{-4} . Richardson extrapolation works here because the trapezoid error has an expansion in even powers of h for smooth integrands; it would not work for the kinked kernel of Section 18.6 without first checking the observed order.

Exercise 18.8. By Lemma 18.1, $I - \lambda T_N$ is invertible on $C[a, b]$ because $I - \lambda \mathbf{A}$ is. Let $g \in C[a, b]$ and $v = (I - \lambda T_N)^{-1}g$, so that

$$v(x) = g(x) + \lambda \sum_{j=1}^N w_j K(x, t_j) v(t_j) \quad \text{for all } x \in [a, b].$$

Setting $x = t_i$ shows that the vector $\mathbf{v} = (v(t_1), \dots, v(t_N))^T$ solves $(I - \lambda \mathbf{A})\mathbf{v} = \mathbf{g}$ with $\mathbf{g} = (g(t_1), \dots, g(t_N))^T$. Hence

$$\|\mathbf{v}\|_\infty \leq \|(I - \lambda \mathbf{A})^{-1}\|_\infty \|\mathbf{g}\|_\infty \leq \|(I - \lambda \mathbf{A})^{-1}\|_\infty \|g\|_\infty.$$

Returning to the formula for $v(x)$,

$$|v(x)| \leq |g(x)| + |\lambda| \sum_{j=1}^N |w_j K(x, t_j)| |v(t_j)| \leq \|g\|_\infty + |\lambda| M_N \|\mathbf{v}\|_\infty.$$

Combining the two estimates and taking the maximum over x ,

$$\|v\|_\infty \leq (1 + |\lambda| M_N \|(I - \lambda \mathbf{A})^{-1}\|_\infty) \|g\|_\infty,$$

which is the claimed bound on the operator norm. For a convergent rule with positive weights, $M_N \leq \max |K| \sum_j w_j = (b - a) \max |K|$ is bounded independently of N , so a bounded matrix inverse gives a bounded stability constant.

Exercise 18.9. For $v \in C[0, 1]$, $(T - T_N)v$ is the constant function with value $\int_0^1 v(t) dt - \sum_j w_j v(t_j)$. Since $|\int_0^1 v| \leq \|v\|_\infty$ and $|\sum_j w_j v(t_j)| \leq \sum_j w_j \|v\|_\infty = \|v\|_\infty$, we get $\|(T - T_N)v\|_\infty \leq 2\|v\|_\infty$, so $\|T - T_N\|_\infty \leq 2$.

For the reverse inequality fix $\varepsilon \in (0, 1)$ and choose $\delta > 0$ so small that the intervals $(t_j - \delta, t_j + \delta)$ are disjoint and $4N\delta < \varepsilon$. Let v be the continuous piecewise-linear function that equals -1 at each node, equals 1 outside the union of these intervals, and is linear on each half-interval $[t_j - \delta, t_j]$ and $[t_j, t_j + \delta]$ (intersected with $[0, 1]$). Then $\|v\|_\infty = 1$, $\sum_j w_j v(t_j) = -1$, and $v \geq -1$ everywhere with $v = 1$ except on a set of length at most $2N\delta$, so

$$\int_0^1 v(t) dt \geq 1 - 2 \cdot 2N\delta > 1 - \varepsilon.$$

Hence $(T - T_N)v > 2 - \varepsilon$ while $\|v\|_\infty = 1$, and $\|T - T_N\|_\infty \geq 2 - \varepsilon$. Since ε was arbitrary, $\|T - T_N\|_\infty = 2$ for every N .

On the other hand, for a fixed v , $\|(T - T_N)v\|_\infty = |\int_0^1 v - Q_N v| \rightarrow 0$ when the rules converge. So $T_N \rightarrow T$ pointwise but not in norm, and the perturbation argument that would use $\|T - T_N\| \rightarrow 0$ is unavailable. This is why Theorem 18.3 needs collective compactness.

Exercise 18.10. (a) Fix j with $1 \leq j \leq m-1$ and put $g(x) = G(x, t_j)$. For $x \leq t_j$, $g(x) = x(1-t_j)$, and for $x \geq t_j$, $g(x) = t_j(1-x)$. So g is linear on $[0, t_j]$ and on $[t_j, 1]$, with slope $1-t_j$ on the left and $-t_j$ on the right, a jump of -1 at $x = t_j$, and $g(0) = g(1) = 0$. The second difference of a linear function vanishes, so for $i \neq j$ (with t_{i-1}, t_i, t_{i+1} all on one side of t_j),

$$g(t_{i+1}) - 2g(t_i) + g(t_{i-1}) = 0.$$

For $i = j$, $g(t_{j+1}) - g(t_j) = -t_j h$ and $g(t_j) - g(t_{j-1}) = (1-t_j)h$, so the second difference is $-t_j h - (1-t_j)h = -h$. The boundary values $g(t_0) = g(t_m) = 0$ take care of the rows $i = 1$ and $i = m-1$. Therefore

$$(\mathbf{LB})_{ij} = \frac{1}{h^2} \left(-hg(t_{i+1}) + 2hg(t_i) - hg(t_{i-1})) \right) = -\frac{1}{h} (g(t_{i+1}) - 2g(t_i) + g(t_{i-1})) = \delta_{ij}.$$

(b) So $\mathbf{B} = \mathbf{L}^{-1}$. The vectors $s_i^{(n)} = \sin(n\pi t_i)$, $n = 1, \dots, m-1$, are eigenvectors of $\mathbf{L}$: from $\sin(\theta + \phi) + \sin(\theta - \phi) = 2 \sin \theta \cos \phi$,

$$-s_{i+1}^{(n)} + 2s_i^{(n)} - s_{i-1}^{(n)} = 2(1 - \cos(n\pi h))s_i^{(n)} = 4 \sin^2\left(\frac{n\pi h}{2}\right)s_i^{(n)},$$

and $s_0^{(n)} = s_m^{(n)} = 0$. Hence $\mathbf{B}$ has eigenvalues $\mu_n = h^2/(4 \sin^2(n\pi h/2))$, and the characteristic values $1/\mu_n$ are $4h^{-2} \sin^2(n\pi h/2) = 4m^2 \sin^2(n\pi/(2m))$. The full Nyström matrix has zero first and last rows (since $G(0, t) = G(1, t) = 0$) and zero first and last columns (since $G(x, 0) = G(x, 1) = 0$), which add only the eigenvalue 0. With $y = n\pi h/2$ and $\sin^2 y = y^2 - \frac{1}{3}y^4 + \mathcal{O}(y^6)$,

$$\frac{4}{h^2} \sin^2\left(\frac{n\pi h}{2}\right) = (n\pi)^2 - \frac{(n\pi)^4 h^2}{12} + \mathcal{O}(h^4).$$

The trapezoid Nyström method for this kernel is therefore the familiar finite-difference method in disguise, and its characteristic values approach $(n\pi)^2$ from below at second order.

Exercise 18.11. The exact integral is $\int_{-h}^h |t| dt = h^2$. Simpson's rule with the three points $-h, 0, h$ gives $\frac{h}{3}(g(-h) + 4g(0) + g(h)) = \frac{h}{3}(2h) = \frac{2}{3}h^2$, so the error is $\frac{1}{3}h^2$.

In row i of the Nyström system the rule is applied to $t \mapsto G(t_i, t)u(t)$, which is smooth except for a corner at $t = t_i$ where the slope jumps by $-u(t_i)$. Write the integrand near t_i as a smooth function plus $-\frac{1}{2}u(t_i)|t-t_i|$. Simpson's composite rule integrates the smooth part with error $\mathcal{O}(h^4)$. If t_i has an odd index, it is the middle point of a double panel $[t_{i-1}, t_{i+1}]$, and on that double panel the corner term contributes an error $-\frac{1}{2}u(t_i) \cdot \frac{1}{3}h^2 = \mathcal{O}(h^2)$. If t_i has an even index, the corner sits on the boundary between double panels, each of which sees a smooth function, and the fourth order survives for that row. The consistency error $\|(T - T_N)u\|_\infty$ is the maximum over all rows, so it is $\mathcal{O}(h^2)$, and so is the Nyström error, as Table 18.2 shows.

Exercise 18.12. For $t \leq x$, $G(x, t) = t(1 - x)$, and for $t \geq x$, $G(x, t) = x(1 - t)$. Hence

$$\int_0^1 G(x, t) dt = (1 - x) \frac{x^2}{2} + x \frac{(1 - x)^2}{2} = \frac{x(1 - x)}{2} (x + (1 - x)) = \frac{x(1 - x)}{2}.$$

At $x = t_i$ the rule applied to $\int_0^1 G(t_i, t)(u(t) - u(t_i)) dt$ gives $\sum_j w_j G(t_i, t_j)(u_j - u_i) = \sum_j A_{ij} u_j - u_i \sum_j A_{ij}$. Adding the exact term $\frac{1}{2} t_i (1 - t_i) u_i$, the discrete equation in row i becomes

$$u_i - \lambda \left(\sum_j A_{ij} u_j - u_i \sum_j A_{ij} + \frac{1}{2} t_i (1 - t_i) u_i \right) = f(t_i),$$

which is $(I - \lambda \mathbf{A}') \mathbf{u} = \mathbf{f}$ with $\mathbf{A}'$ as stated: only the coefficient of u_i in row i changes.

Let $h(t) = G(x, t)(u(t) - u(x))$ with $u \in C^2$. On each side of $t = x$, h is smooth. Its derivative is $h'(t) = \partial_t G(x, t)(u(t) - u(x)) + G(x, t)u'(t)$. As $t \rightarrow x$ from either side, the first term tends to 0 because $u(t) - u(x) \rightarrow 0$ while $\partial_t G$ stays bounded, and the second tends to $G(x, x)u'(x)$. So h' is continuous at $t = x$. The second derivative contains $2 \partial_t G(x, t)u'(t)$, which jumps by $-2u'(x)$ at $t = x$. A jump in the second derivative inside one panel costs a local error of order h^3 , which is the order 3 observed in the last column of Table 18.2.

Exercise 18.13. Since $\lambda_1^{(m)} = 4m^2 \sin^2(\pi/2m)$ increases with m toward π^2 , it is enough to evaluate it for successive m : $\lambda_1^{(41)} \approx 9.86478$ and $\lambda_1^{(42)} \approx 9.865004$. So $m = 42$ is the smallest value, and there the matrix is almost singular at $\lambda = 9.865$. Solving the Nyström system with f built from $u = e^x$ gives largest node errors of about 4.44 for $m = 43$, 0.164 for $m = 64$, 0.0262 for $m = 128$ and 0.0060 for $m = 256$.

For $m < 42$ the discrete characteristic value lies below 9.865, for $m \geq 42$ above it, and near $m = 42$ the matrix $I - \lambda \mathbf{A}$ is nearly singular, which explains the error 4.4 at $m = 43$. Once $\lambda_1^{(m)}$ has moved well past 9.865, the error decreases by factors of about 6 and 4 per doubling, approaching the second-order rate. But the constant is large because the true problem is itself close to resonance: the error carries the factor $|\lambda|/(1 - \lambda/\pi^2)$, which is about 2×10^4 at $\lambda = 9.865$ and about 1.1 at $\lambda = 1$. Near a characteristic value, the mesh must first resolve the characteristic value before it can resolve the solution.

Chapter 19 Numerical Volterra Methods

Exercise 19.1. The kernel is $K(x, t) = x - t$, so $K(x, x) = 0$ and every step is explicit. With $u_0 = 1$ and $x_1 = \frac{1}{2}$,

$$u_1 = 1 + h \left(\frac{1}{2} K(x_1, x_0) u_0 + \frac{1}{2} K(x_1, x_1) u_1 \right) = 1 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = 1.125.$$

At $x_2 = 1$, $K(1, 0) = 1$, $K(1, \frac{1}{2}) = \frac{1}{2}$ and $K(1, 1) = 0$, so

$$u_2 = 1 + \frac{1}{2} \left(\frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot 1.125 + 0 \right) = 1 + \frac{1}{2} \cdot 1.0625 = 1.53125.$$

The exact values are $\cosh \frac{1}{2} \approx 1.127626$ and $\cosh 1 \approx 1.543081$, so the errors are about 0.0026 and 0.0118. The computed values are too small, and the error grows along the march, as the Gronwall argument allows.

Exercise 19.2. Left rectangle rule: $u_n = 1 + \lambda h \sum_{j=0}^{n-1} u_j$ for $n \geq 1$, and $u_0 = 1$. Subtracting the equation for $n - 1$ (for $n \geq 2$) gives $u_n - u_{n-1} = \lambda h u_{n-1}$; for $n = 1$ the equation itself reads $u_1 = 1 + \lambda h u_0 = (1 + \lambda h) u_0$. So $u_n = (1 + \lambda h) u_{n-1}$ for all $n \geq 1$ and $u_n = (1 + \lambda h)^n$. This is the explicit Euler method for $u' = \lambda u$.

Right rectangle rule: $u_n = 1 + \lambda h \sum_{j=1}^n u_j$. Subtracting consecutive equations gives $u_n - u_{n-1} = \lambda h u_n$ for $n \geq 2$, and for $n = 1$, $u_1 = 1 + \lambda h u_1$ gives $u_1 = u_0 / (1 - \lambda h)$. So $u_n = (1 - \lambda h)^{-1} u_{n-1}$ and $u_n = (1 - \lambda h)^{-n}$, provided $\lambda h \neq 1$. This is the implicit Euler method.

Exercise 19.3. Write $z = \lambda h < 0$.

Explicit rectangle: $\rho_E = 1 + z$. Then $|1 + z| < 1$ means $-2 < z < 0$, that is $h < 2/|\lambda|$. And $0 < 1 + z < 1$ means $-1 < z < 0$, that is $h < 1/|\lambda|$. For $1/|\lambda| < h < 2/|\lambda|$ the solution decays but alternates in sign.

Trapezoid: $\rho_T = (1 + z/2)/(1 - z/2)$. For $z < 0$ the denominator $1 - z/2 = 1 + |z|/2$ exceeds the absolute value of the numerator $|1 - |z|/2|$, because $(1 + a)^2 - (1 - a)^2 = 4a > 0$ for $a = |z|/2 > 0$. So $|\rho_T| < 1$ for every h . The factor is positive exactly when $1 + z/2 > 0$, that is $h < 2/|\lambda|$. For larger h it is negative, and it tends to -1 as $h \rightarrow \infty$, so large steps give sign-alternating solutions that decay slowly.

Implicit rectangle: $\rho_I = 1/(1 - z) = 1/(1 + |z|) \in (0, 1)$ for every $h > 0$.

For $\lambda = -50$ these conditions read: explicit needs $h < 0.04$ for decay and $h < 0.02$ for decay without sign changes; trapezoid needs $h < 0.04$ to avoid sign changes; implicit has no restriction. At $h = 0.04$ exactly, $\rho_E = -1$ and $\rho_T = 0$, which are the two special entries in Table 19.3.

Exercise 19.4. For $|z| < 2$, $\log(1 + z/2) - \log(1 - z/2) = 2(\frac{z}{2} + \frac{1}{3}(\frac{z}{2})^3 + \frac{1}{5}(\frac{z}{2})^5 + \dots) = z + \frac{z^3}{12} + \frac{z^5}{80} + \dots$. With $z = \lambda h$ and $N = x/h$,

$$\log u_N = N \log \rho_T = \frac{x}{h} \left(\lambda h + \frac{\lambda^3 h^3}{12} + \mathcal{O}(h^5) \right) = \lambda x + \frac{\lambda^3 x}{12} h^2 + \mathcal{O}(h^4).$$

Exponentiating, $u_N = e^{\lambda x} \exp(\frac{\lambda^3 x}{12} h^2 + \mathcal{O}(h^4)) = e^{\lambda x} (1 + \frac{\lambda^3 x}{12} h^2 + \mathcal{O}(h^4))$, which is the claim. The expansion has only even powers of h , which is why Richardson extrapolation (Chapter 18) works well on these values. For $\lambda = 1$, $x = 1$ the leading term is $e h^2/12$, the prediction column of Table 19.2; for $\lambda = -1$ it is $-e^{-1} h^2/12$, so the computed value is below the exact one.

Exercise 19.5. The nodes are $t_j = jh$, $j = 0, \dots, n-1$ (the diagonal node $t_n = 1$ is skipped), with weight $\frac{h}{2}$ at $j = 0$ and h otherwise. The integrand is $(1 - jh)^{-1/2} = (h(n - j))^{-1/2}$. Hence the value is

$$\frac{h}{2} + h \sum_{j=1}^{n-1} \frac{1}{\sqrt{h(n-j)}} = \frac{h}{2} + \sqrt{h} \sum_{m=1}^{n-1} \frac{1}{\sqrt{m}}.$$

Using the expansion with $n = 1/h$, so that $\sqrt{h} \cdot 2\sqrt{n} = 2$ and $\sqrt{h} \cdot \frac{1}{2} n^{-1/2} = \frac{h}{2}$,

$$\frac{h}{2} + \sqrt{h} \left(2\sqrt{n} + \zeta\left(\frac{1}{2}\right) - \frac{1}{2} n^{-1/2} + \mathcal{O}(n^{-3/2}) \right) = 2 + \zeta\left(\frac{1}{2}\right) \sqrt{h} + \mathcal{O}(h^2).$$

The error is $\zeta(\frac{1}{2})\sqrt{h} \approx -1.4604\sqrt{h}$: order $\frac{1}{2}$. With $n = 16$ this predicts -0.3651 against the actual -0.3653 , and with $n = 256$, -0.09127 in both cases. The mistake is to treat an integrable singularity as a single bad function value. The singular region near $t = 1$ carries a finite share of the integral, about $2\sqrt{h}$ from the last cell alone, and no rule that samples u times the kernel at points can capture it. Product integration, which integrates the kernel exactly, can.

Exercise 19.6. An antiderivative of $(x_n - t)^{-1/2}$ in t is $-2(x_n - t)^{1/2}$. So

$$\int_{x_{j-1}}^{x_j} \frac{dt}{\sqrt{x_n - t}} = 2(\sqrt{x_n - x_{j-1}} - \sqrt{x_n - x_j}) = 2\sqrt{h}(\sqrt{n - j + 1} - \sqrt{n - j}),$$

using $x_n - x_j = (n - j)h$. For $u \equiv 1$ all $u_{j-1/2} = 1$ and

$$2\sqrt{h} \sum_{j=1}^n (\sqrt{n-j+1} - \sqrt{n-j}) = 2\sqrt{h}(\sqrt{n} - \sqrt{0}) = 2\sqrt{nh} = 2\sqrt{x_n},$$

because consecutive terms cancel. So the scheme reproduces the pair $u = 1$, $f = 2\sqrt{x}$ of Benchmark B4 exactly, as it should: the approximation of u by constants is exact for a constant.

Exercise 19.7. Differentiate: $E'(h) = C_1 - \frac{1}{2}C_2\varepsilon h^{-3/2}$, which vanishes at $h_*^{3/2} = C_2\varepsilon/(2C_1)$, that is $h_* = (C_2\varepsilon/(2C_1))^{2/3}$. Since $E'' > 0$ this is the minimum. At h_* , $C_2\varepsilon h_*^{-1/2} = 2C_1 h_*$, so

$$E(h_*) = 3C_1 h_* = 3C_1 \left(\frac{C_2\varepsilon}{2C_1} \right)^{2/3} = \frac{3}{2}(2C_1)^{1/3}(C_2\varepsilon)^{2/3}.$$

Both h_* and the smallest achievable error scale like $\varepsilon^{2/3}$: a data error of 10^{-6} cannot give a better answer than about 10^{-4} , whatever the mesh.

From Table 19.5: the exact-data error at $N = 1024$ is 1.63×10^{-4} , so $C_1 \approx 1.63 \times 10^{-4} \times 1024 \approx 0.17$. The perturbation-driven error at $N = 1024$ is 2.26×10^{-3} with $\varepsilon h^{-1/2} = 10^{-4} \times 32$, so $C_2 \approx 0.71$. Then $h_* = (0.71 \times 10^{-4}/0.34)^{2/3} \approx (2.1 \times 10^{-4})^{2/3} \approx 3.5 \times 10^{-3}$, about $N = 290$, with a total error near $3C_1 h_* \approx 1.8 \times 10^{-3}$. This is consistent with the table, where the combined error is smallest between $N = 64$ and $N = 1024$. The constants are for this particular worst-sign perturbation; random data errors partly cancel and give smaller C_2 .

Exercise 19.8. Differentiating $w(x) = e^{-x} \int_0^x e^t u(t) dt$ gives $w' = -w + u$, and $u = f + \lambda w$, so $w' = (\lambda - 1)w + f$ with $w(0) = 0$. For $f \equiv 1$ and $\lambda = 2$ this is $w' = w + 1$, $w(0) = 0$, so $w = e^x - 1$ and $u = 1 + 2(e^x - 1) = 2e^x - 1$.

Check in the original equation: $\int_0^x e^{-(x-t)}(2e^t - 1) dt = e^{-x}[(e^{2x} - 1) - (e^x - 1)] = e^x - 1$, and $1 + 2(e^x - 1) = 2e^x - 1$.

For a general smooth f the equation has an ODE twin of dimension one, so a standard ODE solver for w gives $u = f + \lambda w$ at cost proportional to the number of steps, with error control built in. Equivalently, the marching method can update the history recursively, because $e^{-(x_n-t)} = e^{-h}e^{-(x_{n-1}-t)}$. The generic $\mathcal{O}(N^2)$ marching of Section 19.1 would work too, but it wastes the structure. When a kernel has such structure, it is wise to use it; when it doesn't, as for $1/(1 + (x - t)^2)$, marching is the method.

Exercise 19.9. With $\lambda = -2$ and $u = x$, the forcing term is $f(x) = x - \lambda(x \arctan x - \frac{1}{2} \log(1 + x^2)) = x + 2(x \arctan x - \frac{1}{2} \log(1 + x^2))$. Trapezoidal marching (19.4) with $h = 2/N$ gives largest grid errors of about 1.846×10^{-3} , 4.593×10^{-4} , 1.147×10^{-4} and

2.868×10^{-5} for $N = 10, 20, 40, 80$. The ratios are about 4.02, 4.00 and 4.00, so the observed order is 2, as Theorem 19.2 predicts. The sign of λ changes the constant, not the order; here $\lambda K(x, x) = -2$ is modest, so no stability restriction appears at these step sizes.

Exercise 19.10. At step n the trapezoid scheme (19.4) uses $K(x_n, x_j)$ for $j = 0, 1, \dots, n$, that is $n + 1$ values. Over N steps the total is $\sum_{n=1}^N (n + 1) = \frac{N(N+1)}{2} + N = \frac{N(N+3)}{2}$ evaluations, and about as many multiply-adds.

For $K(x, t) = a(x)b(t)$ write $b_j = b(x_j)$ and define the partial history

$$S_n = h \left(\frac{1}{2} b_0 u_0 + \sum_{j=1}^{n-1} b_j u_j \right), \quad n \geq 1,$$

so that the right side of (19.4) is $f(x_n) + \lambda a(x_n) S_n$ and the step reads $(1 - \frac{\lambda h}{2} a(x_n) b_n) u_n = f(x_n) + \lambda a(x_n) S_n$. The partial histories satisfy

$$S_1 = \frac{h}{2} b_0 u_0, \quad S_{n+1} = S_n + h b_n u_n \quad (n \geq 1),$$

since going from n to $n + 1$ adds exactly the term $j = n$ with full weight h . Each step therefore costs a fixed number of operations, and the whole march costs $\mathcal{O}(N)$. For a sum of r such terms the cost is $\mathcal{O}(rN)$.

Exercise 19.11. Step n is the scalar equation $c_n u_n = r_n$ with $c_n = 1 - \frac{\lambda h}{2} K(x_n, x_n)$ and r_n the known right side. If $|\lambda| h \max |K| < 2$, then $|\frac{\lambda h}{2} K(x_n, x_n)| < 1$, so $|c_n| \geq 1 - |\frac{\lambda h}{2} K(x_n, x_n)| > 0$ and $u_n = r_n / c_n$ is the unique solution.

For Benchmark B1 with $K \equiv 1$ and $\lambda h = 2$ (for instance $\lambda = 4, h = \frac{1}{2}$), the first step is $(1 - 1)u_1 = 1 + \frac{\lambda h}{2} u_0 = 2$, which has no solution. The amplification factor $\rho_T = (1 + \lambda h/2)/(1 - \lambda h/2)$ has a pole at $\lambda h = 2$, and near it the computed values are huge and meaningless even though the exact solution $e^{\lambda x}$ is perfectly regular. For positive λ this is a real step restriction of the trapezoid rule: $h < 2/(\lambda \max K)$.

Exercise 19.12. The integral is the causal convolution of u with $t^{-1/2}$, and $\frac{1}{\sqrt{\pi}} t^{-1/2} = 1/\sqrt{\pi t}$ has transform $s^{-1/2}$. The transformed equation is therefore $U(s) = \frac{1}{s} - s^{-1/2} U(s)$, so

$$U(s) = \frac{1/s}{1 + s^{-1/2}} = \frac{1}{s + \sqrt{s}} = \frac{1}{\sqrt{s}(\sqrt{s} + 1)},$$

which is the transform of $e^t \operatorname{erfc} \sqrt{t}$. By uniqueness of the Laplace transform for continuous functions of exponential order, $u(x) = e^x \operatorname{erfc} \sqrt{x}$.

With $\lambda = -1/\sqrt{\pi}$, $h = \frac{1}{4}$, $\sqrt{h} = \frac{1}{2}$ and $u_0 = 1$, the product-trapezoid step is

$$(1 - \lambda\sqrt{h} \omega_{1,1})u_1 = 1 + \lambda\sqrt{h} \omega_{1,0}u_0, \quad u_1 = \frac{1 - \frac{1}{2\sqrt{\pi}} \cdot \frac{2}{3}}{1 + \frac{1}{2\sqrt{\pi}} \cdot \frac{4}{3}} = \frac{0.81194}{1.37612} \approx 0.59002.$$

The exact value is $e^{1/4} \operatorname{erfc} \frac{1}{2} \approx 1.284025 \times 0.479500 \approx 0.61569$, so the error of this first step is about 0.026. That is a large error for one step, and it is the first-cell effect seen in Table 19.4: u behaves like $1 - 2\sqrt{x/\pi}$ near 0, and a straight line through u_0 and u_1 cannot follow a square root.

Chapter 20 Collocation and Galerkin Methods

Exercise 20.1. The residual from Example 20.1 is $r_2(x) = (a - 1) + (\frac{b}{2} - a)x + \frac{a}{2}x^2 + \frac{b}{6}x^3$. At $x = 0$ it is $a - 1$, so $a = 1$. At $x = 1$ it is

$$(a - 1) + \frac{b}{2} - a + \frac{a}{2} + \frac{b}{6} = \frac{a}{2} + \frac{2b}{3} - 1,$$

and with $a = 1$ this vanishes for $b = \frac{3}{4}$. Hence $u_2 = 1 + \frac{3}{4}x$. The exact solution at $x = \frac{1}{2}$ is $\cos(\frac{1}{2})/\cos 1 \approx 0.87758/0.54030 \approx 1.6242$, while $u_2(\frac{1}{2}) = 1.375$, an error of about 0.249. The approximation of Example 20.1 gives $u_2(\frac{1}{2}) = (1952 + 768)/1721 \approx 1.5805$, an error of about 0.044 in magnitude at this point. Evaluating on a fine grid, the largest error of $1 + \frac{3}{4}x$ is about 0.255, near $x = 0.55$. Collocating at the ends pins the line to the exact values of $u(0) = 1$ and of the equation at $x = 1$, and leaves the interior, where the solution bends most, unconstrained.

Exercise 20.2. With $u_1 = a$, the residual is $r_1 = a - aT - 1 = a - a(x - \frac{x^2}{2}) - 1$. The Galerkin condition $\int_0^1 r_1 dx = 0$ gives $a - a \cdot \frac{1}{3} - 1 = 0$, since $\int_0^1 (x - \frac{x^2}{2}) dx = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$. Hence $a = \frac{3}{2}$. The exact mean is

$$\int_0^1 \frac{\cos(1-x)}{\cos 1} dx = \frac{\sin 1}{\cos 1} = \tan 1 \approx 1.5574,$$

so the constant approximation differs from the mean of u by about 0.057. Galerkin's constant is not exactly the mean because the orthogonality is imposed on the residual of the equation, not on the error; the two coincide only when $T = 0$.

Exercise 20.3. On $[0, h]$ the 2 hat functions that are nonzero are $1 - t/h$ and t/h . Then $\int_0^h (1 - t/h)^2 dt = h/3$, $\int_0^h (t/h)^2 dt = h/3$ and $\int_0^h (1 - t/h)(t/h) dt = h/6$. An interior hat function is supported on 2 cells, so $M_{ii} = 2h/3$; the end functions live on 1 cell, so $M_{00} = M_{nn} = h/3$. Neighbors overlap on 1 cell, giving $M_{i,i\pm 1} = h/6$, and other pairs do not overlap. The matrix is symmetric, so its eigenvalues are real and its 2-norm condition number is the ratio of the largest to the smallest eigenvalue. Gershgorin's theorem places each eigenvalue in a disc centered at a diagonal entry with radius the sum of the

off-diagonal magnitudes in that row. Interior rows give $[\frac{2h}{3} - \frac{h}{3}, \frac{2h}{3} + \frac{h}{3}] = [\frac{h}{3}, h]$ and end rows give $[\frac{h}{3} - \frac{h}{6}, \frac{h}{3} + \frac{h}{6}] = [\frac{h}{6}, \frac{h}{2}]$. All eigenvalues lie in $[\frac{h}{6}, h]$, so the condition number is at most 6, independent of n . (Computed values approach 4.)

Exercise 20.4. $M_{ij} = \int \varphi_i \varphi_j$ is symmetric. For $\mathbf{K}$,

$$\begin{aligned} K_{ji} &= \int_0^1 \int_0^1 \varphi_j(x) K(x, t) \varphi_i(t) dt dx = \int_0^1 \int_0^1 \varphi_j(t) K(t, x) \varphi_i(x) dx dt \\ &= \int_0^1 \int_0^1 \varphi_i(x) K(x, t) \varphi_j(t) dt dx = K_{ij}, \end{aligned}$$

where the second step renames the variables and the third uses $K(t, x) = K(x, t)$ and Fubini's theorem for the continuous integrand. With real λ , $\mathbf{M} - \lambda \mathbf{K}$ is symmetric. For collocation, the kernel $\min(x, t)$ is symmetric, yet the system in Example 20.1 has matrix entries $\frac{25}{32}, \frac{49}{384}$ in the first row and $\frac{17}{32}, \frac{171}{384}$ in the second, which is not a symmetric matrix. Collocation evaluates the equation at points and has no reason to respect the symmetry of the operator.

Exercise 20.5. The composite trapezoid rule on the mesh is $\int_0^1 g \approx \sum_{k=0}^n w_k g(x_k)$ with $w_0 = w_n = h/2$ and $w_k = h$ otherwise. Applied to $g(t) = K(x_i, t) \varphi_j(t)$ it gives $\sum_k w_k K(x_i, x_k) \varphi_j(x_k) = w_j K(x_i, x_j)$, because $\varphi_j(x_k) = \delta_{jk}$. Also $\varphi_j(x_i) = \delta_{ij}$, so the collocation system (20.5) becomes $c_i - \lambda \sum_j w_j K(x_i, x_j) c_j = f(x_i)$ for $i = 0, \dots, n$. This is the Nyström system of Chapter 18 for the trapezoid rule with unknowns $c_i \approx u(x_i)$, so both methods produce the same nodal values whenever the system is uniquely solvable. Collocation defines u_N between the nodes as the piecewise linear interpolant $\sum_j c_j \varphi_j$. Nyström's method defines it by $u_N(x) = f(x) + \lambda \sum_j w_j K(x, x_j) c_j$, which inherits the smoothness of f and K in x . For problem P the 2 functions agree at the nodes and differ in between.

Exercise 20.6. Write $c = \int_0^1 u(t) dt$. The equation says $u = 1 + cx$, and integrating gives $c = 1 + \frac{c}{2}$, so $c = 2$ and $u = 1 + 2x$. Substituting $u_2 = 2$ into the original equation gives $2 - x \int_0^1 2 dt = 2 - 2x$, which is not 1; the check fails.

The correct Galerkin conditions are $\langle r_2, \varphi_i \rangle = 0$ with $r_2 = u_2 - T u_2 - 1$, so the matrix entries are $\langle \varphi_j, \varphi_i \rangle - \langle T \varphi_j, \varphi_i \rangle$: T acts on the trial function φ_j . With $T1 = x$ and $Tx = \frac{x}{2}$ the correct system is

$$(1 - \frac{1}{2})a + (\frac{1}{2} - \frac{1}{4})b = 1, \quad (\frac{1}{2} - \frac{1}{3})a + (\frac{1}{3} - \frac{1}{6})b = \frac{1}{2},$$

whose solution is $a = 1, b = 2$. Galerkin's method reproduces the exact solution because it lies in the trial space.

The written system uses $\langle \varphi_j, T\varphi_i \rangle = \langle T^*\varphi_j, \varphi_i \rangle$, where T^* has kernel $K^*(x, t) = K(t, x) = t$. It is therefore Galerkin's method for $u - T^*u = 1$, that is $u(x) - \int_0^1 t u(t) dt = 1$. That equation has constant solutions $u = d$ with $d - \frac{d}{2} = 1$, so $u = 2$, which is what the system produced. For a symmetric kernel the slip is harmless; for any other kernel it silently solves the adjoint equation.

Exercise 20.7. The kernel is continuous but its derivative in t jumps at $t = x$. A Gauss rule applied to $t \mapsto |x - t|u(t)$ across the whole interval sees a corner at a point that is generally not a node, so its error decays only algebraically; the Vary step of Example 20.5 shows the same effect for $\min(x, t)$, where Gauss–Legendre Nyström ended up worse than the trapezoid rule. The solution $u = f + \lambda Tu$ is still fairly smooth, but Tu inherits only limited smoothness from the kernel, so global polynomials converge only algebraically as well and give no reward for their cost. Hat functions with moments computed panel by panel, splitting each moment at $t = x_i$ when the corner falls inside a panel, handle the corner exactly in the quadrature and converge at order h^2 by Theorem 20.5. With 60 unknowns that means errors of order 10^{-4} times the size of u'' , predictable and verifiable by halving h . A better alternative still is to split every integral at $t = x$ and use Gauss rules on $[0, x]$ and $[x, 1]$ separately, which restores high-order accuracy; that is a Nyström method adapted to the kernel.

Exercise 20.8. The exact solution satisfies $u = f + \lambda Tu$, and both terms on the right lie in X_N , so $u \in X_N$ and $P_N u = u$ for any projection onto X_N . The error identity (20.10) gives $(I - \lambda P_N T)(u - u_N) = u - P_N u = 0$, where $u - u_N \in X_N$. Suppose $v \in X_N$ satisfies $v - \lambda P_N T v = 0$. Since $T v$ lies in the range of T , which is inside X_N , $P_N T v = T v$, so $v - \lambda T v = 0$ and invertibility of $I - \lambda T$ forces $v = 0$. Hence $u_N = u$, and the projected system is uniquely solvable.

For $u - \int_0^1 x t u(t) dt = 1$, write $c = \int_0^1 t u(t) dt$; then $u = 1 + cx$ and $c = \frac{1}{2} + \frac{c}{3}$, so $c = \frac{3}{4}$ and $u = 1 + \frac{3}{4}x$. The range of T is spanned by x and $f = 1$, both in $\text{span}\{1, x\}$. For instance, collocation at $\frac{1}{3}, \frac{2}{3}$ gives the equations $r_2(\frac{1}{3}) = r_2(\frac{2}{3}) = 0$ with $r_2 = (a - 1) + (b - \frac{a}{2} - \frac{b}{3})x$, and Galerkin's method gives $\int_0^1 r_2 = \int_0^1 x r_2 = 0$; both force $a = 1$ and $b = \frac{3}{4}$, the exact solution.

Exercise 20.9. Build the mesh $x_i = i/n$ for $n = 4, 8, 16, 32, 64$. Compute the moments $\int_0^1 e^{x_i t} \varphi_j(t) dt$ with a Gauss rule of 10 points on each cell, as in the listing, so that quadrature error is far below the discretization error. Solve $(\mathbf{I} - \frac{1}{2}\mathbf{C})\mathbf{c} = \mathbf{f}$, form u_N on the fine grid by `np.interp(xfine, nodes, c)` and take the maximum of $|e^x - u_N(x)|$. The errors are 4.25×10^{-2} , 1.11×10^{-2} , 2.85×10^{-3} , 7.21×10^{-4} and 1.81×10^{-4} . Successive ratios are

close to 4, so the observed orders $\log_2(\text{ratio})$ are 1.93, 1.97, 1.98 and 1.99, approaching the predicted 2. A useful sanity check is that the error is about twice the interpolation error $\frac{h^2}{8} \max|u''| = \frac{e}{8n^2}$, which is consistent with the quasi-optimality bound of Theorem 20.5.

Exercise 20.10. Let $v \in C[0, 1]$ and $g = Tv$. For $x \in [x_{i-1}, x_i]$ write $x = (1 - \theta)x_{i-1} + \theta x_i$ with $\theta \in [0, 1]$; then $P_N g(x) = (1 - \theta)g(x_{i-1}) + \theta g(x_i)$ and

$$|g(x) - P_N g(x)| \leq (1 - \theta)|g(x) - g(x_{i-1})| + \theta|g(x) - g(x_i)| \leq \max_{k=i-1, i} |g(x) - g(x_k)|.$$

For any y with $|x - y| \leq h$,

$$|g(x) - g(y)| = \left| \int_0^1 (K(x, t) - K(y, t))v(t) dt \right| \leq \omega(h) \|v\|_\infty.$$

Taking the maximum over x gives $\|(I - P_N)Tv\|_\infty \leq \omega(h)\|v\|_\infty$, which is the claimed operator bound. If $\partial K/\partial x$ is continuous, the mean value theorem gives $|K(x, t) - K(y, t)| \leq |x - y| \max|\partial_x K|$, so $\omega(h) \leq h \max|\partial_x K|$. For $K = e^{xt}$ on $[0, 1]^2$, $\partial_x K = te^{xt}$ has maximum e , so $\|(I - P_N)T\|_\infty \leq eh$. Uniform continuity of K on the compact square also shows $\omega(h) \rightarrow 0$ without any differentiability, which proves part (a) of Theorem 20.5 for this projection directly.

Exercise 20.11. The kernel $\min(x, t)$ is real, symmetric and continuous, so T is compact and self-adjoint on $L^2(0, 1)$. The functions $e_n(x) = \sqrt{2} \sin((n - \frac{1}{2})\pi x)$ satisfy $Te_n = \mu_n e_n$ with $\mu_n = 1/((n - \frac{1}{2})\pi)^2$ (the Green's function of $-u''$ with $u(0) = 0$, $u'(1) = 0$) and form an orthonormal basis of $L^2(0, 1)$. For $v = \sum v_n e_n$, Parseval's identity gives

$$\langle (I - T)v, v \rangle = \sum_n (1 - \mu_n)|v_n|^2 \geq (1 - \mu_1) \sum_n |v_n|^2 = \left(1 - \frac{4}{\pi^2}\right) \|v\|_2^2,$$

since $\mu_n \leq \mu_1 = 4/\pi^2 \approx 0.405$. So $I - T$ is self-adjoint and positive definite, the hypotheses of Proposition 20.3 hold with $\lambda = 1$, and the Galerkin approximation minimizes $\|u - v\|_E$ over $v \in \text{span}\{1, x\}$. Minimizing the energy error $\|u - a - bx\|_E^2$ directly, as a quadratic function of a and b , gives the same minimizer $a = \frac{156}{139}$, $b = \frac{120}{139}$.

Exercise 20.12. The report attributes to the trial space a failure caused by the projection. With equally spaced points the interpolation projection has exponentially growing norm, so $u - P_N u$ diverges for a function like $1/(1 + 25(2x - 1)^2)$ whose complex singularities lie close to the interval; the error identity (20.10) passes that divergence into the collocation error. The best polynomial approximations of this u converge geometrically, and so does collocation at Chebyshev points: 0.038 at $n = 20$ and 7.1×10^{-4} at $n = 40$ (Figure 20.3). A correct conclusion would be that equally spaced collocation stops improving at $n \approx 8$ and should be replaced by Chebyshev or Gauss points, or by piecewise polynomials on a mesh.

The practice of increasing n until the error stops falling is also risky in general, because it cannot distinguish a projection failure from rounding or from a genuine approximation limit.

Exercise 20.13. Use $p_k(x) = P_k(2x - 1)$, $k = 0, \dots, n - 1$, evaluated with the Legendre routines in NumPy (the function `legval`). Compute $M_{ij} = \int_0^1 p_i p_j$ (which is $\delta_{ij}/(2i + 1)$, a useful check), $K_{ij} = \iint p_i(x)e^{xt}p_j(t)$ and $b_i = \int_0^1 f p_i$ with a composite Gauss rule of 8 panels and 20 points each. Solve $(\mathbf{M} - \frac{1}{2}\mathbf{K})\mathbf{c} = \mathbf{b}$ and evaluate $u_N = \sum c_k p_k$ on 2001 points. The maximum errors are 3.3×10^{-9} for $n = 8$ and 9.7×10^{-11} for $n = 9$, so $n = 9$ is the smallest. Gauss–Legendre Nyström with the natural interpolation formula gives 8.8×10^{-10} with 5 points and 1.6×10^{-12} with 6, so it needs 6. Both converge geometrically, but Nyström’s method uses the kernel at the quadrature points directly, while Galerkin’s approximation is limited by how well a polynomial of degree $n - 1$ approximates e^x in the Galerkin sense. The iterated Galerkin approximation $f + \frac{1}{2}Tu_N$ closes most of that gap.

Chapter 21 First-Kind Equations and Regularization

Exercise 21.1. Each f is twice continuously differentiable with $f(0) = f(1) = 0$, so $u = -f''$. (a) $-(x - x^2)'' = 2$; and $G2 = 2 \cdot \frac{x(1-x)}{2} = x - x^2$, using $G1 = x(1 - x)/2$. (b) $-(\sin \pi x)'' = \pi^2 \sin \pi x$; and $G(\pi^2 \sin \pi t) = \pi^2 \sin(\pi x)/\pi^2 = \sin \pi x$ by Benchmark B3. (c) $-(x^3 - x)'' = -6x$; directly, $G(-6t) = -6[(1 - x) \int_0^x t^2 dt + x \int_x^1 t(1 - t) dt]$. The bracket is $\frac{(1-x)x^3}{3} + x(\frac{1}{6} - \frac{x^2}{2} + \frac{x^3}{3}) = \frac{x}{6} - \frac{x^3}{6}$, so $G(-6t) = x^3 - x$.

Exercise 21.2. Differentiating twice, $-(\varepsilon \sin(n\pi x))'' = \varepsilon(n\pi)^2 \sin(n\pi x)$, so the error in u is $\varepsilon(n\pi)^2 \sin(n\pi t)$. With $\varepsilon = 10^{-6}$ and $n = 100$ its amplitude is $10^{-6} \cdot (100\pi)^2 = 10^{-6} \cdot 98696 \approx 0.0987$. The same error at $n = 1000$ requires $\varepsilon(1000\pi)^2 = 0.0987$, that is $\varepsilon = 0.0987/(9.87 \times 10^6) = 10^{-8}$. The amplification factor $(n\pi)^2$ grows without bound, which is Theorem 21.2 made concrete: the inverse of G is unbounded on $C[0, 1]$ and on $L^2(0, 1)$. A data error too small to see at frequency 1000 produces the same damage as a larger error at frequency 100.

Exercise 21.3. The left side does not depend on x , so a solution can exist only if f is constant; for $f(x) = x$ there is none, and existence fails. If u is a solution, so is $u + v$ for every v with $\int_0^1 v = 0$, such as $v = \sin(2\pi t)$; uniqueness fails. For $f = 3$ the solution set is $\{u : \int_0^1 u = 3\}$, which contains $u = 3$ and $u = 3 + c \sin(2\pi t)$ for every c . Stability in the sense of a bound $\|u\| \leq C\|f\|$ fails as well, since solutions of arbitrarily large norm exist for the same data. The minimum-norm solution in L^2 is the constant $u = 3$, which is what Tikhonov regularization converges to as $\alpha \rightarrow 0$ for this data.

Exercise 21.4. The error in $u = f'$ is $(\delta \sin(\omega x))' = \delta \omega \cos(\omega x)$, with amplitude $\delta \omega = 10^{-3} \cdot 10^4 = 10$. (The perturbed data also violate $f(0) = 0$ unless the perturbation vanishes at 0, which $\sin(\omega x)$ does.) The problem is ill-posed, since the amplification ω is unbounded, but it is mildly ill-posed: one derivative amplifies frequency ω by ω , while the double differentiation of Example 21.1 amplifies by $(n\pi)^2$. The Gaussian blur of Benchmark B6 is far worse than either: its amplification grows like $e^{\sigma^2 \xi^2/2}$, faster than any power, which is called severe ill-posedness.

Exercise 21.5. The matrix is diagonal, so its SVD has $s_1 = 1$, $s_2 = 10^{-3}$ and the standard basis vectors as singular vectors. The direct solution is

$$(1.01/1, (-0.009)/10^{-3}) = (1.01, -9),$$

with error $\|(0.01, -10)\|_2 \approx 10.0$. Truncated SVD with $k = 1$ keeps the first component and discards the second: $(1.01, 0)$, error $\|(0.01, -1)\|_2 \approx 1.00005$. Tikhonov with $\alpha = 10^{-4}$ uses $c_i = s_i f_i^\delta / (s_i^2 + \alpha)$: $c_1 = 1.01/1.0001 \approx 1.0099$ and $c_2 = 10^{-3} \cdot (-0.009)/(10^{-6} + 10^{-4}) \approx -0.0891$. Its error is $\|(0.0099, -1.0891)\|_2 \approx 1.089$. The data's second component is 10^{-3} of signal plus -0.01 of noise; the signal carrying information about u_2 is masked by the larger noise contribution. Truncation sets it to zero; Tikhonov nearly does. Neither can recover $u_2 = 1$, and here the sharp cut of truncation happens to be slightly better than the smooth filter.

Exercise 21.6. For $\mathbf{v} \neq 0$, $\mathbf{v}^\top(A^\top A + \alpha I)\mathbf{v} = \|\mathbf{A}\mathbf{v}\|_2^2 + \alpha\|\mathbf{v}\|_2^2 \geq \alpha\|\mathbf{v}\|_2^2 > 0$, so the symmetric matrix $B = A^\top A + \alpha I$ is positive definite and invertible. Let $\mathbf{u}_\alpha = B^{-1}A^\top \mathbf{f}^\delta$ and write $\mathbf{u} = \mathbf{u}_\alpha + \mathbf{w}$. Expanding,

$$J(\mathbf{u}) = J(\mathbf{u}_\alpha) + 2\mathbf{w}^\top(B\mathbf{u}_\alpha - A^\top \mathbf{f}^\delta) + \mathbf{w}^\top B\mathbf{w} = J(\mathbf{u}_\alpha) + \mathbf{w}^\top B\mathbf{w},$$

and $\mathbf{w}^\top B\mathbf{w} > 0$ unless $\mathbf{w} = 0$. So $\mathbf{u}_\alpha$ is the unique minimizer. At $\alpha = 0$, $B = A^\top A$ is singular when A has a nonzero null vector $\mathbf{z}$; then $J(\mathbf{u} + \mathbf{z}) = J(\mathbf{u})$ for every $\mathbf{u}$, and minimizers are not unique. The penalty $\alpha\|\mathbf{u}\|^2$ is exactly what breaks the tie, by preferring the smallest.

Exercise 21.7. For $s \geq 0$, $(s - \sqrt{\alpha})^2 \geq 0$ gives $s^2 + \alpha \geq 2s\sqrt{\alpha}$, so $s/(s^2 + \alpha) \leq 1/(2\sqrt{\alpha})$, with equality exactly when $s = \sqrt{\alpha}$. In the proof of Theorem 21.7(c), $\mathbf{u}_\alpha - \mathbf{u}_\alpha^0 = \sum_i \frac{s_i}{s_i^2 + \alpha} \mathbf{u}_i^\top (\mathbf{f}^\delta - \mathbf{f}) \mathbf{v}_i$, and orthonormality of the $\mathbf{v}_i$ gives $\|\mathbf{u}_\alpha - \mathbf{u}_\alpha^0\|_2^2 \leq \frac{1}{4\alpha} \|\mathbf{f}^\delta - \mathbf{f}\|_2^2 \leq \frac{\delta^2}{4\alpha}$. As $\alpha \rightarrow 0$ the factor $1/(2\sqrt{\alpha})$ grows without bound, so a small α lets noise through with an amplification approaching $1/s_N$, the amplification of the direct solve.

Exercise 21.8. By Theorem 21.7(b) and orthonormality of the $\mathbf{v}_i$,

$$\|\mathbf{u}_\alpha\|_2^2 = \sum_{i: s_i > 0} \left(\frac{s_i}{s_i^2 + \alpha} \right)^2 (\mathbf{u}_i^\top \mathbf{f}^\delta)^2,$$

and each factor $s_i/(s_i^2 + \alpha)$ is positive and decreasing in α . So $\|\mathbf{u}_\alpha\|_2$ is nonincreasing, and strictly decreasing unless every term with $s_i > 0$ has $\mathbf{u}_i^\top \mathbf{f}^\delta = 0$. Lemma 21.9 says the residual is nondecreasing. Plotting the point (residual, norm) as α increases therefore moves right and down: small α gives large norm and small residual (upper left), large α gives small norm and large residual (lower right). The corner is where the curve turns from steep to flat.

Exercise 21.9. The report treats the residual as a measure of accuracy, which is valid for a well-posed problem with a bounded inverse and invalid here. The exact solution itself has residual about $1.00 \|\eta\|_2$, so any reconstruction with residual $0.66 \|\eta\|_2$ must contain components that explain part of the noise. For truncated SVD those components are the terms $(\mathbf{u}_i^\top \eta)/s_i \mathbf{v}_i$ with small s_i , amplified by factors up to $1/s_{60} \approx 10^{15}$. Table 21.1 shows the consequence: relative error 4.1×10^{11} at $k = 60$, against a minimum of 0.0193 at $k = 17$. The residual decreases monotonically in k , so “smallest residual” always selects the largest k , the worst choice. A correct report would choose k by the discrepancy principle with $\tau = 1.1$, which gives $k = 9$ and error 0.0326, and would state the noise estimate on which that choice rests.

Exercise 21.10. The discrepancy principle is the only rule discussed here with a convergence guarantee, and it needs a noise level, so the task is to estimate that level well rather than to abandon the rule. The data themselves carry an estimate: past the index where the Picard plot levels off, the coefficients $\mathbf{u}_i^\top \mathbf{f}^\delta$ are dominated by noise, and their root-mean-square size estimates $\|\eta\|_2/\sqrt{N}$. Compare that with the manual’s figure. Then apply the discrepancy principle with the better-supported estimate and a safety factor τ slightly above 1. Because underestimating the noise leads to fitting it, the sensitivity check matters: report the reconstruction for the chosen noise level and for twice and half that level, together with α , τ and the residual. The L-curve can be shown as a cross-check, but on Benchmark B6 its corner under-regularizes by a wide margin, so it should not decide the answer.

Exercise 21.11. Build $A_{ij} = K(t_i, t_j)/N$ with $t_j = (j - \frac{1}{2})/N$, $N = 128$; compute exact data with the 4096-point midpoint rule; form η from `default_rng(2026)` scaled to $0.01\|f\|_2$. With the SVD $A = U\Sigma V^\top$, the k -term solution is $\mathbf{u}_k = \sum_{i \leq k} (\mathbf{u}_i^\top \mathbf{f}^\delta/s_i) \mathbf{v}_i$. Scanning k , the relative error is smallest at $k = 17$, where it equals 0.0193. The residual $\|A\mathbf{u}_k - \mathbf{f}^\delta\|_2$ first falls below $1.1\|\eta\|_2 = 0.0958$ at $k = 9$, so the discrepancy principle picks $k = 9$, with error 0.0326. The discrepancy choice is conservative: it stops at the first k that fits the data to within the noise, and the next several terms still carry signal. The loss is less than a factor of 2 in error, which is the price of not knowing u^* .

Exercise 21.12. Let $\mathbf{u}_\alpha^0$ be the Tikhonov solution for exact data $\mathbf{f}$. By Theorem 21.7(c), $\|\mathbf{u}_\alpha - \mathbf{u}_\alpha^0\|_2 \leq \delta/(2\sqrt{\alpha})$. For the bias, write $\mathbf{w} = \sum_i w_i \mathbf{u}_i$ in the left singular vectors, so $\mathbf{u} = A^\top \mathbf{w} = \sum_i s_i w_i \mathbf{v}_i$ and $\mathbf{f} = A\mathbf{u} = \sum_i s_i^2 w_i \mathbf{u}_i$. By Theorem 21.7(b), $\mathbf{u}_\alpha^0 = \sum_i \frac{s_i}{s_i^2 + \alpha} s_i^2 w_i \mathbf{v}_i$, so

$$\mathbf{u} - \mathbf{u}_\alpha^0 = \sum_i \left(s_i - \frac{s_i^3}{s_i^2 + \alpha} \right) w_i \mathbf{v}_i = \sum_i \frac{\alpha s_i}{s_i^2 + \alpha} w_i \mathbf{v}_i.$$

By the bound of Exercise 21.7, $\frac{\alpha s}{s^2 + \alpha} \leq \alpha \cdot \frac{1}{2\sqrt{\alpha}} = \frac{\sqrt{\alpha}}{2}$, so $\|\mathbf{u} - \mathbf{u}_\alpha^0\|_2 \leq \frac{\sqrt{\alpha}}{2} \|\mathbf{w}\|_2$. The triangle inequality gives the stated bound. With $\alpha = \delta/\|\mathbf{w}\|_2$ both terms equal $\frac{1}{2}\sqrt{\delta\|\mathbf{w}\|_2}$, and their sum is $\sqrt{\delta\|\mathbf{w}\|_2}$. The error is of order $\sqrt{\delta}$: halving the noise improves the reconstruction only by a factor of $\sqrt{2}$, and without a source condition no rate at all can be guaranteed.

Exercise 21.13. With $t = x - s$,

$$(k_\sigma \star e^{i\xi \cdot})(x) = \int_{-\infty}^{\infty} k_\sigma(s) e^{i\xi(x-s)} ds = e^{i\xi x} \int_{-\infty}^{\infty} k_\sigma(s) e^{-i\xi s} ds = e^{i\xi x} \hat{k}_\sigma(\xi),$$

and the Fourier transform of the normalized Gaussian is $\hat{k}_\sigma(\xi) = e^{-\sigma^2 \xi^2/2}$ (a standard integral, Chapter 8). Taking imaginary parts, $\sin(\xi t)$ maps to $e^{-\sigma^2 \xi^2/2} \sin(\xi x)$ because $\hat{k}_\sigma$ is real. Setting $e^{-\sigma^2 \xi^2/2} = 0.01$ gives $\sigma^2 \xi^2 = 2 \ln 100 \approx 9.21$, so $\xi \approx 3.035/0.05 \approx 60.7$ and $k = \xi/\pi \approx 19.3$. Modes beyond about the 19th are attenuated below the 1% noise level. Example 21.3 found the best truncation index $k = 17$, and Figure 21.1 shows the noisy coefficients separating from the exact ones in the same range. The whole-line model is only a guide on $[0, 1]$, but it predicts the right place.

Chapter 22 Memory and Hereditary Models

Exercise 22.1. The transform is $\hat{G}(s) = E/(s + 1/\tau)$, so

$$\hat{f}(s) = \frac{s + 1/\tau}{E s^2} = \frac{1}{E} \left(\frac{1}{s} + \frac{1}{\tau s^2} \right),$$

and $J(t) = (1 + t/\tau)/E$. Check: $\int_0^t E e^{-(t-s)/\tau} E^{-1}(1 + s/\tau) ds$. With $r = t - s$ this is $\int_0^t e^{-r/\tau} (1 + (t - r)/\tau) dr$, and a direct integration gives t , as (22.4) requires.

The initial strain $J(0) = 1/E = 1/G(0)$ is the elastic response. After that the strain rate is constant, $\sigma_0/(E\tau)$, because the Maxwell material has no equilibrium modulus ($G(\infty) = 0$): nothing stops it from flowing. The product $E\tau$ plays the role of a viscosity. This is the $E_\infty \rightarrow 0$ limit of Example 22.1, where the retardation time became infinite.

Exercise 22.2. Here $G(t - s) = 1 + e^{-(t-s)}$ and $J(s) = 1 - \frac{1}{2}e^{-s/2}$. Expand the product and integrate term by term over $0 \leq s \leq t$:

$$\begin{aligned} \int_0^t 1 \, ds &= t, \\ -\frac{1}{2} \int_0^t e^{-s/2} \, ds &= -(1 - e^{-t/2}), \\ \int_0^t e^{-(t-s)} \, ds &= 1 - e^{-t}, \\ -\frac{1}{2} \int_0^t e^{-(t-s)} e^{-s/2} \, ds &= -\frac{1}{2} e^{-t} \int_0^t e^{s/2} \, ds = -e^{-t} (e^{t/2} - 1). \end{aligned}$$

The sum is $t - 1 + e^{-t/2} + 1 - e^{-t} - e^{-t/2} + e^{-t} = t$.

Exercise 22.3. If u is dimensionless, u' is in s^{-1} and so is au . The integral $\int_0^t e^{-c(t-s)} u(s) \, ds$ is in seconds, so b times it is in s^{-1} only if b is in s^{-2} . Comparing b with a therefore compares quantities with different units, which is a sign that the stability claim is wrong as well.

The correct comparison is between a and the total memory weight $\int_0^\infty b e^{-cr} \, dr = b/c$, which is a rate. By Example 22.4 and Exercise 22.6, all solutions decay exactly when $b < ac$. For example, $a = 1, c = 3, b = 2$ satisfies $b > a$ but is stable (Example 22.3), while $a = 1, c = \frac{1}{4}, b = \frac{1}{2}$ satisfies $b < a$ but has $ac - b = -\frac{1}{4} < 0$ and is unstable.

Exercise 22.4. If ε' is continuous on $[0, \infty)$, then $G(t-s)\varepsilon'(s)$ is bounded on $0 \leq s \leq t \leq 1$, say by M , and

$$\left| \int_0^t G(t-s)\varepsilon'(s) ds \right| \leq Mt \rightarrow 0 \quad (t \rightarrow 0^+).$$

The left side is the constant $\sigma_0 > 0$, so the equation fails for small t and cannot have such a solution.

The mistake is the assumption that the strain starts continuously from 0. A material with a finite instantaneous modulus responds to a sudden stress with a sudden strain $\varepsilon(0^+) = \sigma_0/G(0)$, and the Boltzmann integral must include the response to that jump:

$$\sigma_0 = G(t)\varepsilon(0^+) + \int_0^t G(t-s)\varepsilon'(s) ds.$$

At $t = 0^+$ this gives $\varepsilon(0^+) = \sigma_0/G(0)$, as it should. Equivalently, use the integrated-by-parts form (22.2), which leads to the Volterra equation (22.3) and involves no derivative of ε . For the standard linear solid its solution is $\varepsilon = \sigma_0 J$ with J from (22.5).

Exercise 22.5. The proof of Proposition 22.1 works with $c = 0$ (then $w = \int_0^t u$ and $w' = u$), giving $u'' + au' - bu = 0$, $u(0) = 1$, $u'(0) = -a$. With $a = 1$, $b = 2$: $u'' + u' - 2u = 0$ has roots 1 and -2 , so $u = Ae^t + Be^{-2t}$ with $A + B = 1$ and $A - 2B = -1$. Hence $B = \frac{2}{3}$, $A = \frac{1}{3}$ and $u = \frac{1}{3}(e^t + 2e^{-2t})$.

Check in the original equation: $u' = \frac{1}{3}(e^t - 4e^{-2t})$ and $\int_0^t u = \frac{1}{3}(e^t - 1) - \frac{1}{3}(e^{-2t} - 1) = \frac{1}{3}(e^t - e^{-2t})$. Then

$$-u + 2 \int_0^t u = -\frac{1}{3}e^t - \frac{2}{3}e^{-2t} + \frac{2}{3}e^t - \frac{2}{3}e^{-2t} = \frac{1}{3}e^t - \frac{4}{3}e^{-2t} = u'.$$

For general $b > 0$ the product of the roots r_1, r_2 of $s^2 + as - b$ is $-b < 0$, so $r_1 > 0 > r_2$. The initial conditions give the coefficient of the growing exponential as $u_0(-a - r_2)/(r_1 - r_2) = u_0 r_1/(r_1 - r_2)$, using $r_1 + r_2 = -a$, and this is nonzero whenever $u_0 \neq 0$. So the solution grows for every $b > 0$. An infinite memory with positive feedback always wins in the end, in agreement with the criterion $b < ac$, which fails when $c = 0$.

Exercise 22.6. By Proposition 22.1 the solutions are the solutions of $u'' + pu' + qu = 0$ with $p = a + c > 0$ and $q = ac - b$, for all initial values $u(0) = u_0$ (and $u'(0) = -au_0$). Let r_1, r_2 be the roots of $s^2 + ps + q$, so $r_1 + r_2 = -p$ and $r_1 r_2 = q$.

If $q > 0$: for complex roots, $\operatorname{Re} r_{1,2} = -p/2 < 0$. For real roots, their product is positive, so they have the same sign, and their sum is $-p < 0$, so both are negative. Every solution is a combination of $e^{r_1 t}$ and $e^{r_2 t}$ (or e^{rt} and te^{rt} for a double root $r = -p/2 < 0$), and all of these tend to 0.

If $q \leq 0$: the roots are real (the discriminant $p^2 - 4q > 0$) and one of them is ≥ 0 . The solution with $u_0 = 1$ is $u = Ae^{r_1 t} + Be^{r_2 t}$ with $r_1 \geq 0 > r_2$ and $A = (-a - r_2)/(r_1 - r_2)$. Since $r_2 = -p - r_1 = -a - c - r_1 < -a$, we have $-a - r_2 > 0$ and $A > 0$, so u does not tend to 0. (For $q = 0$ it tends to $A > 0$, as in Example 22.4 with $b = 3$.)

So every solution decays iff $ac - b > 0$, that is, $b < ac$. Dividing by c , the condition reads $b/c < a$: the total weight $\int_0^\infty be^{-cr} dr = b/c$ that the memory feeds back must be smaller than the rate a at which u decays on its own.

Exercise 22.7. The roots of $s^2 + (a+c)s + (ac - \kappa c)$ are $r_\pm = \frac{1}{2}(-(a+c) \pm \sqrt{(a-c)^2 + 4\kappa c})$. For large c ,

$$\sqrt{(c-a)^2 + 4\kappa c} = (c-a)\sqrt{1 + \frac{4\kappa c}{(c-a)^2}} = (c-a) + 2\kappa + \mathcal{O}(1/c),$$

so $r_+ = \frac{1}{2}(-a-c+c-a+2\kappa) + \mathcal{O}(1/c) \rightarrow \kappa - a$ and $r_- = -c + \mathcal{O}(1) \rightarrow -\infty$. The fast mode dies out almost instantly and the slow mode is $e^{(\kappa-a)t}$, the solution of $u' = -(a-\kappa)u$.

The reason is visible in the integral: $\kappa c e^{-c(t-s)}$ is a weight of total mass $\kappa(1 - e^{-ct}) \rightarrow \kappa$ concentrated in the last $\mathcal{O}(1/c)$ seconds, so for continuous u the integral tends to $\kappa u(t)$. A very short memory acts like an instantaneous feedback of strength κ .

Exercise 22.8. Transforming, $s\hat{u} - u_0 = -a\hat{u} + b\hat{u}\left(\frac{1}{s+c_1} + \frac{1}{s+c_2}\right)$. Multiplying by $(s+c_1)(s+c_2)$,

$$\hat{u} \left[(s+a)(s+c_1)(s+c_2) - b((s+c_2) + (s+c_1)) \right] = u_0(s+c_1)(s+c_2),$$

so $P(s) = (s+a)(s+c_1)(s+c_2) - b(2s+c_1+c_2)$, a cubic. Its three roots are the exponents in u , so u satisfies a third-order linear ODE with characteristic polynomial P .

In the time domain, put $w_i(t) = \int_0^t e^{-c_i(t-s)} u(s) ds$. Exactly as in the proof of Proposition 22.1, $w'_i = u - c_i w_i$ with $w_i(0) = 0$, and the model becomes $u' = -au + b(w_1 + w_2)$. Each exponential in the kernel costs one extra variable. A kernel that is a sum of N exponentials therefore gives a system of $N+1$ first-order equations; this is how long-memory kernels are often approximated in practice, by fitting a sum of exponentials over the time range of interest.

Exercise 22.9. By Example 22.2 with $\beta = \frac{1}{2}$, $J(t) = \frac{\sin(\pi/2)}{(\pi/2)G_0}(t/t_0)^{1/2} = \frac{2}{\pi G_0}(t/t_0)^{1/2}$. Since t/t_0 is dimensionless and G_0 is in pascals, J is in inverse pascals, as a compliance must be. The strain under a stress σ_0 is $\varepsilon = \sigma_0 J$, dimensionless. Check with the Beta integral: $\int_0^t (t-s)^{-1/2} s^{1/2} ds = t B(\frac{1}{2}, \frac{3}{2}) = \pi t/2$, and $G_0 \cdot \frac{2}{\pi G_0} \cdot \frac{\pi}{2} t = t$.

Exercise 22.10. Let $w(t) = \int_0^t \tau^{-1} e^{-(t-s)/\tau} h(s) ds$, the sensor's weighted average. Then $w(0) = 0$ and, by the Leibniz rule, $w' = (h - w)/\tau$. The model is $h' = q - kw$, so $h'(0) = q$ and $h'' = -kw' = -k(h - w)/\tau$. Using $kw = q - h'$,

$$h'' = -\frac{k}{\tau}h + \frac{1}{\tau}(q - h'), \quad \text{that is} \quad h'' + \frac{1}{\tau}h' + \frac{k}{\tau}h = \frac{q}{\tau}.$$

The constant particular solution is $h = q/k$, the equilibrium level at which the averaged outflow balances the inflow. The characteristic polynomial $s^2 + s/\tau + k/\tau$ has positive coefficients, so both roots have negative real parts and $h \rightarrow q/k$. The roots are complex, and the approach is oscillatory, exactly when the discriminant is negative:

$$\frac{1}{\tau^2} - \frac{4k}{\tau} < 0 \iff 4k\tau > 1.$$

The physical reading is that a sensor with a long averaging time τ or a strong response k keeps correcting for levels that no longer exist, so it overshoots. Checking in the original equation for a sample case ($k = 2$, $\tau = \frac{1}{2}$, $q = h_0 = 1$, so $4k\tau = 4$) by computing the convolution numerically at $t = 1.4$ and $t = 3$ gives residuals below 10^{-12} .

Exercise 22.11. The exact value is $J(10) = 1 - \frac{1}{2}e^{-5} \approx 0.996631$. The scheme is

$$\varepsilon_n = \frac{1}{2} + \frac{h}{2} \left(\frac{1}{2} e^{-t_n} \varepsilon_0 + \sum_{j=1}^{n-1} e^{-(t_n - t_j)} \varepsilon_j + \frac{1}{2} \varepsilon_n \right), \quad \varepsilon_0 = \frac{1}{2},$$

solved for ε_n at each step. The computed values at $t = 10$ are about 0.999937, 0.997456 and 0.996837 for $h = 0.2, 0.1$ and 0.05 , with errors 3.31×10^{-3} , 8.25×10^{-4} and 2.06×10^{-4} . The ratios of successive errors are 4.01 and 4.00, so the error behaves like Ch^2 : the trapezoidal rule with a smooth kernel is second-order accurate, as Chapter 19 proves.

Exercise 22.12. By (22.4), $s\hat{J}(s) = 1/(s\hat{G}(s))$. For the standard linear solid,

$$s\hat{G}(s) = E_\infty + \frac{E_1 s}{s + 1/\tau} \rightarrow E_\infty \quad (s \rightarrow 0^+),$$

so $J(\infty) = 1/E_\infty$, in agreement with (22.5). In general, if $s\hat{G}(s) \rightarrow G(\infty) > 0$, then $s\hat{J}(s) \rightarrow 1/G(\infty)$, and the final value theorem gives $J(\infty) = 1/G(\infty)$. The theorem needs a hypothesis: the limit of J must exist, which holds, for example, when J is monotone and bounded. For the hyperbolic kernel of Section 22.5 the same argument predicts $J(\infty) = 1$, which the computed values in Table 22.2 approach.

Exercise 22.13. The transform of $t^{-\beta}$ is $\Gamma(1 - \beta)s^{\beta-1}$, so transforming the model gives $s\hat{u} - u_0 = -a\hat{u} + b\Gamma(1 - \beta)s^{\beta-1}\hat{u}$ and

$$\hat{u}(s) = \frac{u_0}{s + a - b\Gamma(1 - \beta)s^{\beta-1}}.$$

If u solved a linear constant-coefficient ODE of order N , transforming that ODE would give $\hat{u} = Q(s)/P(s)$ with polynomials P and Q . Suppose $\hat{u} = Q/P$. Then, with $u_0 \neq 0$,

$$b \Gamma(1 - \beta) s^{\beta-1} = s + a - \frac{u_0 P(s)}{Q(s)},$$

so $s^{\beta-1}$ would be a rational function $R(s)$ on $s > 0$. But a rational function behaves like Cs^m with an integer m as $s \rightarrow \infty$, while $s^{\beta-1}$ has the non-integer exponent $\beta - 1$. This contradiction shows that no finite-order ODE reduction exists. The exponential kernel of Proposition 22.1 works precisely because its transform $1/(s + c)$ is rational.

Chapter 23 Boundary Integral Formulations

Exercise 23.1. Write $r^2 = |x - y|^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2$, so $\Phi = -\frac{1}{4\pi} \log r^2$. Then

$$\partial_{x_i} \Phi = -\frac{1}{4\pi} \cdot \frac{2(x_i - y_i)}{r^2} = -\frac{x_i - y_i}{2\pi r^2}, \quad \partial_{x_i}^2 \Phi = -\frac{1}{2\pi} \left(\frac{1}{r^2} - \frac{2(x_i - y_i)^2}{r^4} \right).$$

Summing over $i = 1, 2$ gives $-\frac{1}{2\pi} \left(\frac{2}{r^2} - \frac{2r^2}{r^4} \right) = 0$ for $r > 0$. Since Φ depends on $x - y$ only, $\partial_{y_i} \Phi = -\partial_{x_i} \Phi = \frac{x_i - y_i}{2\pi r^2}$. Hence $\nabla_y \Phi = \frac{x - y}{2\pi |x - y|^2}$ and $\partial \Phi / \partial \nu(y) = \nabla_y \Phi \cdot \nu(y) = \frac{(x - y) \cdot \nu(y)}{2\pi |x - y|^2}$.

Exercise 23.2. For $x = (\rho, 0)$ and $y = (\cos t, \sin t)$, $\nu(y) = y$, $(x - y) \cdot y = \rho \cos t - 1$ and $|x - y|^2 = \rho^2 - 2\rho \cos t + 1$. Write $\rho \cos t - 1 = -\frac{1}{2}(\rho^2 - 2\rho \cos t + 1) + \frac{1}{2}(\rho^2 - 1)$. Then

$$\int_0^{2\pi} \frac{\rho \cos t - 1}{2\pi(\rho^2 - 2\rho \cos t + 1)} dt = -\frac{1}{2} + \frac{\rho^2 - 1}{4\pi} \int_0^{2\pi} \frac{dt}{\rho^2 + 1 - 2\rho \cos t}.$$

With $A = \rho^2 + 1$ and $B = 2\rho$, $\int_0^{2\pi} \frac{dt}{A - B \cos t} = \frac{2\pi}{\sqrt{A^2 - B^2}} = \frac{2\pi}{|1 - \rho^2|}$. For $\rho = \frac{1}{2}$ the second term is $\frac{\rho^2 - 1}{4\pi} \cdot \frac{2\pi}{1 - \rho^2} = -\frac{1}{2}$, so the total is -1 . For $\rho = 2$ it is $\frac{\rho^2 - 1}{4\pi} \cdot \frac{2\pi}{\rho^2 - 1} = \frac{1}{2}$, so the total is 0 . Both agree with Theorem 23.3.

Exercise 23.3. The mean of g over a period is $\bar{g} = 1$, so (23.10) gives

$$\varphi(t) = -2g(t) + \bar{g} = -2 - 2\cos t - 2\sin 2t + 1 = -1 - 2\cos t - 2\sin 2t.$$

Check in the boundary equation: $\int_0^{2\pi} \varphi d\tau = -2\pi$, so $K\varphi = -\frac{1}{4\pi}(-2\pi) = \frac{1}{2}$ and $-\frac{1}{2}\varphi + K\varphi = \frac{1}{2} + \cos t + \sin 2t + \frac{1}{2} = g$. By Example 23.3 the potential $\mathcal{D}\varphi$ equals the Poisson integral of g , which is the harmonic function with these boundary values. In polar coordinates $r^n \cos nt$ and $r^n \sin nt$ are harmonic, so $u = 1 + r \cos t + r^2 \sin 2t = 1 + x_1 + 2x_1x_2$. It equals g on $r = 1$ and is harmonic, so by uniqueness it is the solution.

Exercise 23.4. Fix $x \notin \bar{D}$. Since $\bar{D}$ is compact and x lies outside it, $\text{dist}(x, \bar{D}) = d > 0$, and $y \mapsto \Phi(x, y)$ is C^∞ on the open set $\{y : |y - x| > d/2\}$, which contains $\bar{D}$. Harmonicity in y follows from Exercise 23.1 with the roles of x and y exchanged, because Φ is symmetric.

The divergence theorem holds on the bounded domain D with C^2 boundary for the C^1 vector field $F(y) = \nabla_y \Phi(x, y)$, and gives

$$\int_{\partial D} F \cdot \nu \, ds = \int_D \operatorname{div} F \, dy = \int_D \Delta_y \Phi(x, y) \, dy = 0.$$

The left side is $\int_{\partial D} \partial \Phi(x, y) / \partial \nu(y) \, ds(y)$.

Exercise 23.5. On the unit circle $K\varphi = -\frac{1}{4\pi} \int_0^{2\pi} \varphi \, d\tau$. The proposed equation reads $\frac{1}{2}\varphi(t) - \frac{1}{4\pi} \int_0^{2\pi} \varphi \, d\tau = 1$. Integrating over $t \in [0, 2\pi]$ gives

$$\frac{1}{2} \int_0^{2\pi} \varphi \, dt - \frac{2\pi}{4\pi} \int_0^{2\pi} \varphi \, d\tau = 2\pi,$$

that is $0 = 2\pi$, a contradiction. So the equation has no solution. But the interior Dirichlet problem with $g \equiv 1$ has the obvious solution $u \equiv 1$, and Theorem 23.7 guarantees a unique density for the correct equation (here $\varphi \equiv -1$). The reversed sign has produced the *exterior* operator $\frac{1}{2}I + K$, whose null space on the unit circle contains the constants. Testing a boundary equation on the constant data and Gauss's lemma is a cheap way to catch such sign errors.

Exercise 23.6. With $n(\tau) = (b \cos \tau, a \sin \tau) = \nu|z'|$, $\tilde{k}(t, \tau) = \frac{(z(t)-z(\tau)) \cdot n(\tau)}{2\pi|z(t)-z(\tau)|^2}$. The numerator is

$$ab \cos \tau (\cos t - \cos \tau) + ab \sin \tau (\sin t - \sin \tau) = ab(\cos(t - \tau) - 1) = -2ab \sin^2 \frac{t - \tau}{2}.$$

By the half-angle identities, $|z(t) - z(\tau)|^2 = a^2 \cdot 4 \sin^2 \frac{t + \tau}{2} \sin^2 \frac{t - \tau}{2} + b^2 \cdot 4 \cos^2 \frac{t + \tau}{2} \sin^2 \frac{t - \tau}{2}$. Cancelling $\sin^2 \frac{t - \tau}{2}$ gives (23.11). At $\tau = t$ the kernel is $-\frac{ab}{4\pi(a^2 \sin^2 t + b^2 \cos^2 t)}$. Since $|z'(t)| = (a^2 \sin^2 t + b^2 \cos^2 t)^{1/2}$, the product $\kappa|z'|$ equals $ab/(a^2 \sin^2 t + b^2 \cos^2 t)$, so the diagonal value is $-\kappa|z'|/(4\pi)$.

Exercise 23.7. On the ellipse $u = x_2$ has boundary values $g(t) = b \sin t = \sin t$. Its only Fourier coefficient is $\beta_1 = 1$. With $q = \frac{a-b}{a+b} = \frac{1}{3}$, formula (23.14) gives $\varphi(t) = -\frac{2}{1-1/3} \sin t = -3 \sin t$. Check with (23.13): $K \sin t = \frac{1}{6} \sin t$, so $-\frac{1}{2}\varphi + K\varphi = \frac{3}{2} \sin t - \frac{1}{2} \sin t = \sin t = g$. By Theorem 23.7, $\mathcal{D}\varphi$ is the solution of the Dirichlet problem with data g , which is x_2 .

Exercise 23.8. First, Symm's equation (23.17) is of the first kind. On a circle of radius R , (23.18) shows that S has eigenvalues $R/(2n)$ on the modes $\cos nt$ and $\sin nt$. They tend to zero, so S^{-1} is unbounded and errors in the data are amplified by factors of order n in the

n th mode; a discretization inherits condition numbers that grow with N . Second, uniqueness can fail: on the unit circle $S1 = -R \log R = 0$, so constants lie in the null space and the constant data $g = 1$ cannot be matched at all. The double-layer equation $-\frac{1}{2}\varphi + K\varphi = g$ is of the second kind and uniquely solvable on every smooth closed curve (Theorem 23.7); on the ellipse with $a = 2$, $b = 1$ the inverse has norm 3 and the Nyström matrices have condition number 3 for every N in Example 23.5. The colleague's observation is true but it is the wrong criterion: avoiding the jump relation costs a well-posed equation.

Exercise 23.9. By (23.18), $S \cos t = \frac{R}{2} \cos t$, so $\psi = \frac{2}{R} \cos t$ solves (a). Also $S1 = -R \log R$, so the constant $\psi = -1/(R \log R)$ solves (b) whenever $R \log R \neq 0$, that is $R \neq 1$ (the radius is positive). At $R = 1$ we have $S1 = 0$ and the equation $S\psi = 1$ has no solution: the mean of $S\psi$ is $-R \log R$ times the mean of ψ , which is 0, while the mean of the data is 1. The null space is spanned by the constant: every other mode is multiplied by $R/(2n) \neq 0$, so $S\psi = 0$ forces all nonconstant Fourier coefficients of ψ to vanish.

Exercise 23.10. Since $0 \leq q < 1$, the series $\sum q^n \cos n(t + \tau)$ is dominated by $\sum q^n < \infty$ and converges uniformly on $[0, 2\pi]^2$, so it may be integrated term by term against any bounded function. Against 1: every cosine term integrates to 0, leaving $K1 = -\frac{1}{4\pi} \cdot 2\pi = -\frac{1}{2}$. Against $\cos m\tau$: $\cos n(t + \tau) \cos m\tau = \cos nt \cos n\tau \cos m\tau - \sin nt \sin n\tau \cos m\tau$ integrates to $\pi \cos nt$ if $n = m$ and to 0 otherwise, so $K \cos mt = -\frac{1}{4\pi} \cdot 2q^m \pi \cos mt = -\frac{1}{2} q^m \cos mt$. Against $\sin m\tau$: the same expansion gives $-\pi \sin mt$ for $n = m$, so $K \sin mt = \frac{1}{2} q^m \sin mt$. The functions $1, \cos nt, \sin nt$ form a complete orthogonal system in $L^2(0, 2\pi)$, and $-\frac{1}{2}I + K$ acts diagonally on it with eigenvalues $-1, -\frac{1}{2}(1 + q^n)$ and $-\frac{1}{2}(1 - q^n)$. None vanishes, and their moduli are bounded below by $\frac{1}{2}(1 - q)$, attained at $\sin t$. For a diagonal operator in an orthonormal basis the inverse has norm equal to the reciprocal of the smallest modulus, $2/(1 - q)$. With $a = 2, b = 1, q = \frac{1}{3}$ and the norm is 3.

Exercise 23.11. On the unit circle the equation is $\frac{1}{2}\varphi - \frac{1}{4\pi} \int_0^{2\pi} \varphi d\tau = g$. Write $I = \int_0^{2\pi} \varphi d\tau$. Integrating over a period gives $\frac{1}{2}I - \frac{2\pi}{4\pi}I = \int_0^{2\pi} g dt$, that is $0 = \int_0^{2\pi} g dt$. So mean-zero data are necessary. Conversely, if $\int g = 0$, every $\varphi = 2g + c$ with a constant c is a solution: $\int \varphi = 2\pi c$, so $K\varphi = -\frac{c}{2}$ and $\frac{1}{2}\varphi + K\varphi = g + \frac{c}{2} - \frac{c}{2} = g$. The homogeneous equation has the constants as solutions and no others (for mean-zero φ it reduces to $\frac{1}{2}\varphi = 0$). This is the Fredholm alternative of Chapter 12 at work: the kernel is constant and symmetric, the adjoint null space is also spanned by 1, and solvability requires g orthogonal to it. The potential explains why: $\mathcal{D}\varphi(x) = \mathcal{O}(1/|x|)$ as $|x| \rightarrow \infty$, while the

exterior Dirichlet problem for data with nonzero mean has a bounded solution tending to a nonzero constant at infinity. The standard cure adds a constant or a point-source term to the ansatz [41].

Exercise 23.12. On this ellipse $g = a^2 \cos^2 t - b^2 \sin^2 t = \frac{a^2 - b^2}{2} + \frac{a^2 + b^2}{2} \cos 2t = 0.625 + 1.625 \cos 2t$ and $q = \frac{0.5}{2.5} = 0.2$, so (23.14) gives $\varphi = -0.625 - \frac{2 \cdot 1.625^2}{1.04} \cos 2t = -0.625 - 3.125 \cos 2t$. The Nyström system is built exactly as in the Computation box with $a = 1.5$. The computed errors are:

N	largest density error at the nodes	error at $(0, 0)$	error at $(1, 0.3)$
8	1.9×10^{-4}	2.4×10^{-2}	1.1×10^{-1}
16	5.0×10^{-10}	3.8×10^{-5}	8.9×10^{-3}
32	1.8×10^{-15}	9.8×10^{-11}	3.0×10^{-5}

The density converges fastest, because the only error in it comes from the quadrature of the smooth kernel. The potential at the center converges next, and the point $(1, 0.3)$, which is closer to the boundary, converges most slowly, for the reason explained in the Watch Out box after Example 23.5. Digits in the last place may differ slightly with a different implementation.

Exercise 23.13. By (23.11) the kernel $\tilde{k}$ is continuous (even analytic) on $[0, 2\pi]^2$. An integral operator with continuous kernel on a compact interval maps bounded sets of $C[0, 2\pi]$ to bounded, equicontinuous sets, so it is compact by the Arzelà–Ascoli theorem (Chapter 11). Its norm on $C[0, 2\pi]$ is $\max_t \int_0^{2\pi} |\tilde{k}(t, \tau)| d\tau$. Since $\tilde{k} < 0$ everywhere and $\int_0^{2\pi} \tilde{k}(t, \tau) d\tau = -\frac{1}{2}$ for every t (the constant term of (23.12), or Gauss’s lemma on the curve), $\|K\|_\infty = \frac{1}{2}$ exactly. For S the kernel $-\frac{1}{2\pi} \log|z(t) - z(\tau)||z'(\tau)|$ is unbounded on the diagonal, so the continuous-kernel argument does not apply. Compactness still holds, because the singularity is weak, but that requires the weakly singular kernel theory of Chapter 15 or an approximation argument.

Chapter 24 From Data to a Defensible Solution

Exercise 24.1. (a) With $A = U\Sigma V^T$ for a square matrix, $A^T A = V\Sigma^2 V^T$ and $A^T f^\delta = V\Sigma U^T f^\delta$. So $(A^T A + \alpha I)u_\alpha = A^T f^\delta$ becomes $V(\Sigma^2 + \alpha I)V^T u_\alpha = V\Sigma U^T f^\delta$. Multiplying by V^T and solving the diagonal system, the i th coefficient of u_α in the basis v_i is $s_i(u_i^T f^\delta)/(s_i^2 + \alpha)$, which is (24.2) after writing $\frac{s_i}{s_i^2 + \alpha} = \frac{s_i^2}{s_i^2 + \alpha} \cdot \frac{1}{s_i}$. (b) Since $s_i^2 + \alpha > 0$, the inequality $\frac{s_i^2}{s_i^2 + \alpha} \geq \frac{1}{2}$ is equivalent to $2s_i^2 \geq s_i^2 + \alpha$, that is $s_i \geq \sqrt{\alpha}$. (c) $\sqrt{5.97 \times 10^{-3}} = 0.0773$. The singular values are decreasing, $s_{15} = 0.0867 \geq 0.0773$ and $s_{16} = 0.0622 < 0.0773$, so exactly the first 15 components have filter factors of at least $\frac{1}{2}$. This agrees with the Picard plot, where the noisy coefficients reach the noise level near $i = 15$.

Exercise 24.2. By (24.2), $Au_\alpha = \sum_i s_i \cdot \frac{s_i}{s_i^2 + \alpha} c_i u_i = \sum_i \frac{s_i^2}{s_i^2 + \alpha} c_i u_i$, while $f^\delta = \sum_i c_i u_i$ because the u_i form an orthonormal basis. Hence $Au_\alpha - f^\delta = -\sum_i \frac{\alpha}{s_i^2 + \alpha} c_i u_i$ and, by orthonormality, $\rho(\alpha)^2 = \sum_i \left(\frac{\alpha}{s_i^2 + \alpha}\right)^2 c_i^2$. For $s_i > 0$ the function $\alpha \mapsto \alpha/(s_i^2 + \alpha)$ has derivative $s_i^2/(s_i^2 + \alpha)^2 > 0$, so each term with $c_i \neq 0$ is strictly increasing, and at least one such term exists because $c \neq 0$. So ρ is continuous and strictly increasing. As $\alpha \rightarrow 0$ every factor tends to 0, so $\rho \rightarrow 0$; as $\alpha \rightarrow \infty$ every factor tends to 1, so $\rho^2 \rightarrow \sum c_i^2 = \|f^\delta\|_2^2$. If $0 < \tau\|\eta\|_2 < \|f^\delta\|_2$, the intermediate value theorem gives some α with $\rho(\alpha) = \tau\|\eta\|_2$, and strict monotonicity makes it unique.

Exercise 24.3. For any $v \neq 0$, $v^T(A^T A + \alpha I)v = \|Av\|_2^2 + \alpha\|v\|_2^2 > 0$, so $A^T A + \alpha I$ is symmetric positive definite, hence invertible and the normal equations have exactly one solution. From (24.2), the coefficient of u_α along v_i is $\frac{s_i}{s_i^2 + \alpha} c_i$. By the arithmetic-geometric mean inequality $s_i^2 + \alpha \geq 2s_i\sqrt{\alpha}$, so $\frac{s_i}{s_i^2 + \alpha} \leq \frac{1}{2\sqrt{\alpha}}$. Therefore $\|u_\alpha\|_2^2 = \sum_i \left(\frac{s_i}{s_i^2 + \alpha}\right)^2 c_i^2 \leq \frac{1}{4\alpha} \sum_i c_i^2 = \frac{\|f^\delta\|_2^2}{4\alpha}$. For the benchmark, $\|f^\delta\|_2 = 8.713$ and $\sqrt{\alpha} = 0.0773$, so the bound is 56.4, about 6 times the actual 8.89. The bound is attained only if all of f^δ lies in a component with $s_i = \sqrt{\alpha}$; here most of f^δ lies in components with s_i near 1, whose factors are close to 1.

Exercise 24.4. (a) Put $t = x - s$. Then $\sin(\omega t) = \sin(\omega x) \cos(\omega s) - \cos(\omega x) \sin(\omega s)$. The Gaussian $g(s) = \frac{1}{\sigma\sqrt{2\pi}} e^{-s^2/(2\sigma^2)}$ is even, so $\int g(s) \sin(\omega s) ds = 0$, and its characteristic function is $\int g(s) \cos(\omega s) ds = e^{-\sigma^2\omega^2/2}$ (the Fourier transform of a Gaussian is a Gaussian). Hence the convolution equals $e^{-\sigma^2\omega^2/2} \sin(\omega x)$. (b) We need $e^{-\sigma^2 k^2 \pi^2/2} \geq 0.01$, that is $k^2 \leq \frac{2 \log 100}{\sigma^2 \pi^2} = \frac{9.2103}{0.024674} = 373.3$, so $k \leq 19.32$ and the largest integer is $k = 19$, with factor 0.0116; for $k = 20$ the factor is 0.0072. This calculation gives an attenuation threshold on the whole line. On the finite interval the singular vectors are only approximately sines, and recoverability depends on their signal and noise coefficients. The threshold therefore does not count recoverable components; the Picard plot and reconstruction checks supply that evidence.

Exercise 24.5. The residual measures how well u satisfies the discrete equations with the noisy data, not how close u is to u^* . The data contain noise with $\|\eta\|_2 \approx 0.044$ at this grid size, and the exact solution does *not* satisfy $Au^* = f^\delta$: its residual is about the size of the noise. A residual far below the noise level is therefore a warning sign, because it means the computed u has fitted the noise as well. Error and residual are linked by $u - u^* = A^{-1}((Au - f^\delta) + (f^\delta - Au^*))$, and the second term has size about $\|\eta\|_2$; multiplied by A^{-1} , whose norm is about 10^5 here, it produces an error of about 740 in absolute size, which is 1.65×10^2 times $\|u^*\|_2 = 4.47$. The correct comparison is the one the discrepancy principle makes: a trustworthy reconstruction has a residual close to $\|\eta\|_2$, not smaller. The distinction between residual and error is the one drawn in Chapter 4, and first-kind problems are where it matters most.

Exercise 24.6. A real instrument produces data from the continuous equation. Any discretization adds a modeling error $Au^* - f$ that the inversion cannot distinguish from noise. When the test data are made with the same matrix, this error is exactly zero, so the test sees a problem that is easier than the real one: the method only has to cope with the added noise, and parameter rules calibrated on such data can choose α too small. Computing the exact data with a much finer rule, as the benchmark does, puts the modeling error back and lets you measure it. The relative modeling errors $\|Au^* - f\|_2 / \|f\|_2$ are 4.25×10^{-3} ($N = 16$), 1.01×10^{-3} ($N = 32$), 2.48×10^{-4} ($N = 64$) and 6.17×10^{-5} ($N = 128$). Ten percent of the noise level is 10^{-3} . At $N = 32$ the modeling error is right at that threshold (1.01×10^{-3} , just above it), so $N = 64$ is the smallest grid in the list that is safely below it. At $N = 16$ the modeling error is about 43% of the noise and would have to be included in the noise estimate used by the discrepancy principle.

Exercise 24.7. (a) Write $u_\alpha - u^* = b + n$ with the bias $b = u_\alpha^f - u^*$ (the reconstruction from exact data minus the truth) and the noise part $n = u_\alpha - u_\alpha^f$. These are vectors, and the triangle inequality only gives $\|b + n\|_2 \leq \|b\|_2 + \|n\|_2 = 0.248$. If b and n were orthogonal

the total would be $\sqrt{0.127^2 + 0.121^2} = 0.175$; the actual 0.167 is slightly smaller, so the two parts are close to orthogonal and even cancel a little. Error budgets for vectors combine like this, not by adding norms. (b) Each of the 100 reconstructions in the uncertainty study has the same bias (to the extent that α changes little between seeds) and a different noise part. The band therefore describes the spread of the noise part around the biased reconstruction. It cannot detect the bias itself, which shifts every realization in the same direction. That is why the band misses 0 near the ends of the interval, where the bias is largest: the band is narrow and in the wrong place. A defensible report states both, the spread from the band and the bias from a manufactured test.

Exercise 24.8. The TSVD residual is $\|\sum_{i>k} c_i u_i\|_2 = (\sum_{i>k} c_i^2)^{1/2}$, where $c_i = u_i^T f^\delta$. For $k = 9$ the noise in the 119 discarded components has norm 0.082 (about $0.0077\sqrt{119}$), while the signal they contain is concentrated in components 11, 13, 15 and 17, with exact coefficients 0.051, 0.023, 0.0095 and 0.0036 and combined norm 0.057. Signal and noise mix in each coefficient, and the resulting residual at $k = 9$ is $1.034\|\eta\|_2 = 0.090$, already below the threshold $1.1\|\eta\|_2 = 0.096$. The rule stops, although components 11 to 17 still carry signal larger than their noise. The discrepancy principle judges the fit by one number, the residual norm, and a signal component smaller than the total noise norm is invisible to it. Tikhonov regularization with the same rule chooses α so that $\sqrt{\alpha} \approx 0.077$, and the smooth filter keeps components 10 to 15 at more than half strength and components beyond with reduced weight. It does not have to decide sharply where the signal ends, so the loss from the rule is much smaller (1.87% against the best 1.80%).

Exercise 24.9. By Exercise 24.2, $\rho(\alpha)^2 = \sum_i \left(\frac{\alpha}{s_i^2 + \alpha}\right)^2 c_i^2$. For components with $s_i^2 \ll \alpha$ the factor is almost 1, and for $s_i^2 \gg \alpha$ it is almost 0, so $\rho(\alpha)^2$ is roughly the sum of c_i^2 over the components with $s_i < \sqrt{\alpha}$. Beyond $i \approx 15$ these coefficients are noise of size about 0.0077. So if m components have $s_i < \sqrt{\alpha}$, then $\rho(\alpha) \approx 0.0077\sqrt{m}$. With the true noise norm the target 0.096 is reached with $\alpha \approx 6 \times 10^{-3}$, where many small components are filtered. The underestimated target 0.0479 requires $m \approx (0.0479/0.0077)^2 \approx 39$, so only 39 of the 128 components may be filtered and about 89 must be kept. The singular values fall below 10^{-15} from $i = 60$ on, so keeping 89 components means $\sqrt{\alpha}$ must lie below singular values of size 10^{-16} or less, that is α of order 10^{-32} . That is also why ρ is so flat: across 26 decades of α only the few components whose s_i cross $\sqrt{\alpha}$ change, and each changes ρ^2 by only 0.0077^2 . Once components with $s_i \approx 10^{-16}$ are kept, their noise coefficients of size 0.0077 are divided by s_i , giving contributions of order 10^{13} to 10^{14} , which is the error observed. The lesson for the report is that the discrepancy principle is only as reliable as the noise estimate it is given, and an underestimate fails catastrophically, not gradually.

Exercise 24.10. With the functions of the benchmark module, A is the 256×256 midpoint matrix, the exact data are $f = A_{\text{fine}} u^*$ sampled at the 256 midpoints, and $f^\delta = f + \eta$ with η scaled to $0.01 \|f\|_2$. The discrepancy equation gives $\alpha = 5.01 \times 10^{-3}$ and $\|u_\alpha - u^*\|_2 / \|u^*\|_2 = 1.63\%$. The noise realization is different from the $N = 128$ one (it has 256 components), so the comparison mixes a grid change with a noise change. The error lies inside the range 1.29% to 2.81% found over 100 seeds at $N = 128$, and the chosen α is close to the $N = 128$ values. Refining the grid therefore neither improves nor damages the result noticeably, which supports the statement that the noise and the regularization bias, not the discretization, limit the accuracy.

Exercise 24.11. With 16 midpoints the discrete operator only contains components with at most about 16 sign changes, and its smallest singular value is still about 1/11 of the largest, so the noise is amplified by at most a factor of about 11. Discretization has played the role of truncation: the components that would have destroyed the solve are not there. This is called regularization by discretization. It is not a defensible method here for three reasons. First, the grid size acts as a regularization parameter chosen without any rule tied to the noise level; with $N = 32$ the same approach gives an error of 1.65×10^2 (Exercise 24.5), so the answer depends on a choice that nobody justified. Second, the coarse grid carries a modeling error of about 43% of the noise level, which enters the result uncontrolled. Third, the resulting error of 6.3% is more than three times that of Tikhonov regularization with the discrepancy principle on the fine grid. A parameter chosen by a stated rule, on a grid fine enough that the model error is negligible, is the defensible version of the same idea.

Exercise 24.12. Tikhonov regularization is linear in the data: $u_\alpha = R_\alpha f^\delta$ with $R_\alpha = (A^T A + \alpha I)^{-1} A^T$. So $u_\alpha - u^* = (R_\alpha f - u^*) + R_\alpha \eta$, and the triangle inequality gives $\|u_\alpha - u^*\|_2 \leq \|u_\alpha^f - u^*\|_2 + \|R_\alpha \eta\|_2$. By the argument of Exercise 24.3, every coefficient of $R_\alpha \eta$ is the corresponding coefficient $u_i^T \eta$ multiplied by $s_i / (s_i^2 + \alpha) \leq 1 / (2\sqrt{\alpha})$, so $\|R_\alpha \eta\|_2 \leq \|\eta\|_2 / (2\sqrt{\alpha})$. For the benchmark this is $0.08707 / (2 \times 0.0773) = 0.563$, while the actual noise part is 0.121. The bound is pessimistic because it treats every noise coefficient as if it were multiplied by the largest possible factor $1 / (2\sqrt{\alpha})$, which happens only for $s_i = \sqrt{\alpha}$. In fact the noise is spread evenly over all 128 components; those with large s_i are multiplied by about $1/s_i \leq 1/0.3$ and those with small s_i by about s_i/α , which is tiny. Only a handful of components near $s_i \approx \sqrt{\alpha}$ receive the worst factor. The bound is still useful: it is guaranteed, and it shows that the noise part grows like $\alpha^{-1/2}$ as α decreases while the bias shrinks, which is the trade-off behind Figure 24.3.

Exercise 24.13. A possible revision: “The claim assumes the instrument width is known to within $\pm 20\%$; within that range the relative error stays below 4% (at worst 3.98% when the true width is underestimated by 20% and 3.46% when it is overestimated by 20%), so an error bound of 2 to 3% should not be quoted without a better calibration of σ .” The other sentences are not affected by this change, but two of them deserve a remark. The statement about the ends of the interval remains true, and a wrong σ changes the size of the bias there, so the region not to be interpreted should be stated as “within about one width of either end” rather than as a fixed 0.05. The precondition about the noise estimate is unchanged, because the collapse under an underestimated noise level does not depend on σ . A defensible report updates exactly the claims the new information touches and says so.

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