

GEOMETRY

The Pythagorean Theorem

The theorem, its converse, triples, the distance formula, 3D diagonals, and a 10-question practice set with full solutions.

SUBJECT	LEVEL	INCLUDES
Mathematics	High school and early college	Practice set, answer key, revision sheet

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INSIDE

- 1 Three proofs compared
- 2 Verified triples and wrong-answer diagnostics
- 3 The Statement
- 4 Worked Examples
- 5 Geometric Proof: Square Rearrangement
- 6 Algebraic Proof: Similar Triangles
- 7 Pythagorean Triples
- 8 Historical Reference Card for Triples
- 9 The Converse
- 10 Generalization: The Law of Cosines
- 11 Generalization: Higher Dimensions
- 12 Pythagorean Identity in Trigonometry
- 13 Where the Pythagorean Theorem Shows Up
- 14 Common Mistakes With the Pythagorean Theorem
- 15 A Brief History of the Pythagorean Theorem
- 16 Generalization: Pythagorean Theorem in Inner Product Spaces
- 17 Pythagorean Theorem in Machine Learning

18 Worked Example: 3D Distance

19 Geometric Proofs Beyond the Standard

20 Pythagorean Theorem and the Distance Formula

21 Pythagorean Theorem in Computer Graphics

22 Reverse Engineering: Finding Sides From Legs

23 Three Common Misconceptions

24 Practice Questions

25 Answer Key and Revision Sheet

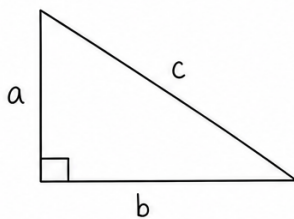
The **Pythagorean theorem** says that in any right triangle, the square of the hypotenuse equals the sum of the squares of the other two sides. Algebraically: $a^2 + b^2 = c^2$. It's one of the oldest, most-proved, and most-used results in mathematics, over 350 distinct proofs have been published, and the theorem appears in every culture's mathematical history independently.

Despite being 2,500+ years old and provable from elementary geometry, the Pythagorean theorem still earns its place in modern math. Distance formulas in any number of dimensions are direct generalizations. Trigonometry is built on it. Norms in linear algebra inherit the same structure. Anywhere distance, length, or hypotenuse-style relationships matter, the Pythagorean theorem is doing the work in the background.

THE PYTHAGOREAN THEOREM

In a right triangle, the sum of the squares of the legs equals the square of the hypotenuse.

1. RIGHT TRIANGLE



2. THE THEOREM

$$a^2 + b^2 = c^2$$

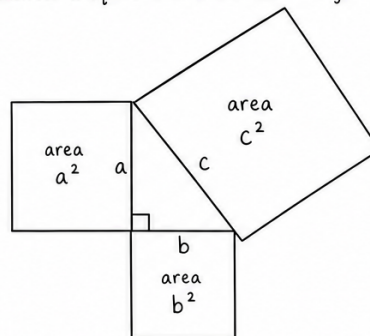
a = length of one leg

b = length of the other leg

c = length of the hypotenuse
(opposite the right angle)

3. GEOMETRIC PROOF

Construct a square on each side of the right triangle.



The two leg-squares fit together perfectly to form the hypotenuse-square.

$$\begin{array}{|c|} \hline \text{area} \\ a^2 \\ \hline \end{array} + \begin{array}{|c|} \hline \text{area} \\ b^2 \\ \hline \end{array} = \begin{array}{|c|} \hline \text{area} \\ c^2 \\ \hline \end{array}$$

$$\text{Therefore, } a^2 + b^2 = c^2$$

4. EXAMPLE: 3-4-5 TRIANGLE

Let $a = 3$, $b = 4$, $c = 5$.

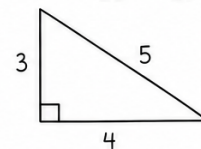
Check:

$$a^2 + b^2 = c^2$$

$$3^2 + 4^2 = 5^2$$

$$9 + 16 = 25$$

$$25 = 25 \quad \checkmark$$



So a 3-4-5 triangle is a right triangle.

COMMON MULTIPLES:

Multiply 3-4-5 by the same number to get other Pythagorean triples.

(6-8-10), (9-12-15), (12-16-20), ...

Pythagorean theorem right triangle squares geometric proof modern textbook illustration

Three proofs compared

The theorem can be proved in many ways. These three routes emphasize different ideas, so the best proof depends on what you are trying to learn.

PROOF	MAIN TOOL	WHAT IT REVEALS	COMMON STICKING POINT
Square rearrangement	Area	Four triangles leave a square of area c^2	Accounting for the inner square
Similar triangles	Proportion	Each leg is a geometric mean	Matching corresponding sides
Trapezoid area	Two area formulas	The identity follows from one composite shape	Setting the two area expressions equal

Verified triples and wrong-answer diagnostics

Every row below was checked by direct substitution into $a^2 + b^2 = c^2$. Primitive and scaled triples are both useful, but the hypotenuse must be the longest side.

A	B	C	CHECK
3	4	5	$9 + 16 = 25$
5	12	13	$25 + 144 = 169$
8	15	17	$64 + 225 = 289$
7	24	25	$49 + 576 = 625$
20	21	29	$400 + 441 = 841$
12	35	37	$144 + 1,225 = 1,369$
9	40	41	$81 + 1,600 = 1,681$
28	45	53	$784 + 2,025 = 2,809$

- **Wrong:** $a + b = c$. **Fix:** square the side lengths before adding.
- **Wrong:** $a^2 + b^2 = c$. **Fix:** take the positive square root only after finding c^2 .
- **Wrong:** using a leg as c . **Fix:** label the side opposite the right angle as the hypotenuse.
- **Wrong:** applying the theorem to any triangle. **Fix:** use it only for a right triangle, or use the converse

to test whether the triangle is right.

- **Wrong:** rounding too early. **Fix:** keep the radical or full precision until the final answer.

The Statement

For a right triangle with legs of lengths a and b and hypotenuse c :

$$a^2 + b^2 = c^2$$

The hypotenuse is the side opposite the right angle, always the longest of the three sides. The other two sides (the legs) are the ones that form the right angle.

The theorem is reversible: any triangle satisfying $a^2 + b^2 = c^2$ is a right triangle. This is the converse, and it's just as useful as the forward direction, it lets you test whether three given side lengths form a right triangle without measuring angles.

Worked Examples

Find the hypotenuse: A right triangle has legs 3 and 4. Then $c^2 = 9 + 16 = 25$, so $c = 5$. This is the famous 3-4-5 triangle, used since antiquity for laying out right angles in construction.

Find a missing leg: A right triangle has hypotenuse 13 and one leg 5. Then $b^2 = 13^2 - 5^2 = 169 - 25 = 144$, so $b = 12$. Another classic Pythagorean triple.

Distance between two points: The distance between $(1, 2)$ and $(4, 6)$ is $\sqrt{(4-1)^2 + (6-2)^2} = \sqrt{9+16} = 5$. The distance formula in 2D is just the Pythagorean theorem applied to the horizontal and vertical separations.

Geometric Proof: Square Rearrangement

One of the most elegant proofs uses a square of side $a + b$. Inside the square, arrange four copies of the right triangle (with legs a, b , hypotenuse c) so that they leave a smaller square of side c in the middle.

The total area of the big square is $(a + b)^2 = a^2 + 2ab + b^2$. The same area equals the four triangles plus the inner square: $4 \cdot \frac{1}{2}ab + c^2 = 2ab + c^2$. Setting them equal: $a^2 + 2ab + b^2 = 2ab + c^2$, so $a^2 + b^2 = c^2$.

This proof is purely visual and works without any algebraic manipulation. There are dozens of variations using different rearrangements, all leading to the same conclusion.

Algebraic Proof: Similar Triangles

Drop a perpendicular from the right angle to the hypotenuse. This divides the original right triangle into two smaller triangles, both similar to the original.

Let the foot of the perpendicular split the hypotenuse into segments of length p (adjacent to leg a) and q (adjacent to leg b). By similar triangles: $a/c = p/a$, so $a^2 = pc$. Similarly $b^2 = qc$.

Adding: $a^2 + b^2 = pc + qc = c(p + q) = c \cdot c = c^2$. The proof is short, uses only similarity arguments, and reveals the deep structural reason behind the Pythagorean relationship.

Pythagorean Triples

A Pythagorean triple is three positive integers (a, b, c) with $a^2 + b^2 = c^2$. The classic example is $(3, 4, 5)$. Other primitive triples include $(5, 12, 13)$, $(8, 15, 17)$, $(7, 24, 25)$, and $(20, 21, 29)$.

Euclid's formula generates every primitive Pythagorean triple from coprime integers $m > n$ of opposite parity:

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2$$

Triples appear in surveying, construction, and any context where you need to lay out right angles using

only a measuring tape. The 3-4-5 carpenter’s triangle is the practical workhorse, knot a string at intervals of 3, 4, and 5 units, pull tight, and you have a guaranteed right angle.

Historical Reference Card for Triples

A handful of triples are worth memorizing because they appear so often in textbook problems and in real-world layout: (3, 4, 5), (5, 12, 13), (8, 15, 17), (7, 24, 25), (9, 40, 41), and their multiples.

Once you can recognize a Pythagorean triple by sight, many problems collapse to inspection rather than computation. Geometry exam questions often use these specifically because they avoid messy square roots in the answer.

PYTHAGOREAN THEOREM · INTEGER-ONLY TRIANGLES

Pythagorean triples worth memorizing

Right triangles where all three sides are whole numbers. Spotting these on sight saves work in geometry, surveying, and construction layout.

TRIPLE	VERIFICATION	TYPE	COMMON USE
3, 4, 5	$9 + 16 = 25$	Primitive	The "carpenter's triangle"
5, 12, 13	$25 + 144 = 169$	Primitive	Tile and surveying layouts
8, 15, 17	$64 + 225 = 289$	Primitive	Right-angle confirmation
7, 24, 25	$49 + 576 = 625$	Primitive	Trigonometry exam staples
9, 40, 41	$81 + 1600 = 1681$	Primitive	Less common, still useful
6, 8, 10	$36 + 64 = 100$	$2\times (3, 4, 5)$	Construction layouts
9, 12, 15	$81 + 144 = 225$	$3\times (3, 4, 5)$	Larger-scale layouts
20, 21, 29	$400 + 441 = 841$	Primitive	Number theory examples

Primitive triples have no common factor across all three sides. Every other Pythagorean triple is an integer multiple of a primitive one. Euclid's formula ($m^2-n^2, 2mn, m^2+n^2$) generates all primitives from coprime integers $m > n$.

Pythagorean triples reference table with verification and use cases

The Converse

The converse of the Pythagorean theorem says: if a triangle has sides a, b, c with $a^2 + b^2 = c^2$, then it's a right triangle (with the right angle opposite the side c).

This converse is what makes the theorem usable as a test. You don't need to measure angles directly, measure side lengths and check the Pythagorean equality. Builders, surveyors, and machinists use the converse routinely.

The converse extends to a trichotomy. If $a^2 + b^2 > c^2$, the triangle is acute (all angles less than 90°). If $a^2 + b^2 < c^2$, the triangle is obtuse (one angle greater than 90°). If equal, it's a right triangle. This trichotomy generalizes to the law of cosines.

Generalization: The Law of Cosines

For any triangle (not just right triangles) with sides a, b, c and the angle C opposite side c :

$$c^2 = a^2 + b^2 - 2ab \cos C$$

This is the law of cosines. It reduces to the Pythagorean theorem when $C = 90^\circ$ (since $\cos 90^\circ = 0$). The law of cosines lets you analyze any triangle, not just right ones, and it's the standard tool in trigonometry for solving general triangles.

Generalization: Higher Dimensions

The distance between two points in 3D space follows directly from the Pythagorean theorem applied twice:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The same pattern extends to n -dimensional space. The Euclidean norm of a vector $(v_1, v_2, \dots, v_n)$ is

$\sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$. This generalization is the basis for distance, length, and norm in linear algebra and machine learning. Whenever you compute a Euclidean distance in any dimension, you're applying the Pythagorean theorem.

Pythagorean Identity in Trigonometry

For any angle θ :

$$\sin^2 \theta + \cos^2 \theta = 1$$

This is the Pythagorean identity, and it's the Pythagorean theorem applied to the unit circle. The point $(\cos \theta, \sin \theta)$ lies on the unit circle (radius 1), so the sum of the squares of its coordinates equals 1.

The Pythagorean identity is the foundation for nearly every other trig identity. Other useful forms (dividing through by $\sin^2 \theta$ or $\cos^2 \theta$) give $1 + \tan^2 \theta = \sec^2 \theta$ and $\cot^2 \theta + 1 = \csc^2 \theta$. Read more in the related [trigonometric identities](#) note.

Where the Pythagorean Theorem Shows Up

Geometry and trigonometry: distance formulas, the Pythagorean identity, the law of cosines, vector norms. **Construction and surveying:** right-angle layout using string knotted at 3-4-5 intervals. **Computer graphics:** distance calculations between points, collision detection, ray length computations. **Physics:** magnitude of vector quantities (force, velocity, momentum, electric field) computed from components. **Navigation:** straight-line distance from horizontal and vertical (or east-west and north-south) separations. **Machine learning:** Euclidean distance in feature space, L2 norm regularization, k-nearest-neighbor algorithms. **Engineering:** stress and strain analysis, beam deflection, vibration mode amplitudes.

Common Mistakes With the Pythagorean Theorem

Applying it to non-right triangles. The theorem only works for right triangles. For other triangles, use the law of cosines. **Confusing legs and hypotenuse.** The hypotenuse is opposite the right angle and is always the longest side. c in the formula must be the hypotenuse; a and b are the legs. **Forgetting to take the square root.** $c^2 = a^2 + b^2$ gives you c^2 , not c . Don't stop at the squared value when the problem asks for the length. **Negative answers.** Distances are positive. If your algebra produces a negative root, take the positive value (or recheck the problem setup). **Mixing units.** All three sides must be in the same unit. Mixing meters and feet (or inches and centimeters) produces nonsense. **Misidentifying the right angle.** Without a right angle, the theorem doesn't apply. Verify the angle is 90° before using the formula.

A Brief History of the Pythagorean Theorem

Hundreds of proofs have been published, using rearrangement, similar triangles, algebra, calculus, and other methods. One well-known area proof was published by James A. Garfield before he became a U.S. president. The variety matters more than an exact proof count because collections use different rules for deciding whether two variants are distinct.

Generalization: Pythagorean Theorem in Inner Product Spaces

The Pythagorean theorem extends to any inner product space. For orthogonal vectors u and v :

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

where $\|\cdot\|$ is the norm derived from the inner product. This generalizes to function spaces, signal spaces, and abstract Hilbert spaces, the foundation of Fourier analysis, signal processing, and quantum mechanics. The familiar 2D version is the simplest example of a result that holds across the entire mathematical universe of inner product spaces.

Pythagorean Theorem in Machine Learning

Euclidean distance in feature space, used by k-nearest-neighbors, k-means clustering, t-SNE, UMAP, and many other algorithms, is the Pythagorean theorem applied to high-dimensional vectors. The L2 norm in regularization (ridge regression) is the same calculation.

Cosine similarity, the standard alternative for high-dimensional embedding spaces, doesn't use Pythagorean distance directly but is closely related, it normalizes vectors first, then dots them. Read more in the related notes on [dot and cross products](#) and [vectors](#).

Worked Example: 3D Distance

Find the distance between points (1, 2, 3) and (4, 6, 11) in 3D space. Apply the 3D distance formula:

$$d = \sqrt{(4-1)^2 + (6-2)^2 + (11-3)^2} = \sqrt{9 + 16 + 64} = \sqrt{89} \approx 9.43$$

The 3D version is the Pythagorean theorem applied twice, once for the horizontal projection (giving $\sqrt{9+16} = 5$), then once more combining that with the vertical separation ($\sqrt{5^2 + 8^2} = \sqrt{89}$). This decomposition extends to any number of dimensions.

Geometric Proofs Beyond the Standard

Three of the most elegant proofs of the Pythagorean theorem:

Over 350 distinct proofs of the Pythagorean theorem exist, more than any other theorem in mathematics. Each illuminates a different aspect of why the relationship holds.

Pythagorean Theorem and the Distance Formula

The 2D distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is exactly the Pythagorean theorem applied to the horizontal and vertical separations between two points. Form a right triangle with the line segment between the points as the hypotenuse and axis-aligned legs as the other two sides; the distance follows directly. Most coordinate-geometry problems collapse to this single formula plus its 3D extension. Understanding that the distance formula *is* the Pythagorean theorem (not a separate result) makes everything from simple geometry homework to vector norms in machine learning instantly more intuitive.

Pythagorean Theorem in Computer Graphics

3D graphics engines compute distances between points constantly, for collision detection, lighting calculations, level-of-detail decisions, and culling objects too far from the camera. Each distance computation is the Pythagorean theorem applied in 3D: $d = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$.

Performance-critical code often skips the square root and compares squared distances directly: if you only need to know which of two points is closer, comparing d_1^2 and d_2^2 gives the same answer as comparing d_1 and d_2 , and avoids the costly sqrt operation. This shortcut is everywhere in game engines, physics simulations, and ray tracing, a small optimization that adds up across millions of distance comparisons per frame.

Reverse Engineering: Finding Sides From Legs

Given a right triangle's hypotenuse and one leg, the other leg is $\sqrt{c^2 - a^2}$. For $c = 25$, $a = 7$: $b = \sqrt{625 - 49} = \sqrt{576} = 24$. The triangle is 7-24-25, a primitive Pythagorean triple.

This kind of reverse computation appears in surveying (computing perpendicular distances from observed angles and slopes), construction (verifying square corners by measuring diagonals), and computer graphics (computing missing components of vectors).

The general lesson: if any two sides of a right triangle are known, the third follows from the Pythagorean theorem. Memorizing the common triples lets you skip the arithmetic when the numbers fit a known

pattern.

Three Common Misconceptions

First: the Pythagorean theorem doesn't apply to all triangles. It requires a right angle (exactly 90°). For other triangles, use the law of cosines.

Second: c in the formula is always the hypotenuse, never a leg. The hypotenuse is the side opposite the right angle and is always the longest side. Mixing up which side is which gives wrong answers.

Third: the formula gives c^2 , not c . Forgetting to take the square root at the end is the most common arithmetic mistake on Pythagorean problems.

Practice Questions

Work each question before reading its solution. The set runs from direct recall and substitution to the applied questions that exams actually use to separate grades.

Q1. A right triangle has legs 6 and 8. Find the hypotenuse.

SOLUTION

$c = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$. This is the 3-4-5 triple scaled by 2.

Q2. A right triangle has hypotenuse 13 and one leg 5. Find the other leg.

SOLUTION

$b = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$. Subtract when the hypotenuse is known; adding here is the classic error.

Q3. Is a triangle with sides 7, 24, 25 right-angled?

SOLUTION

Check the converse: $7^2 + 24^2 = 49 + 576 = 625 = 25^2$. Yes, the angle opposite the 25 side is exactly 90° . The converse turns the theorem into a test.

Q4. A 5 m ladder leans against a wall with its base 1.4 m from the wall. How high does it reach?

SOLUTION

$h = \sqrt{5^2 - 1.4^2} = \sqrt{25 - 1.96} = \sqrt{23.04} = 4.8$ m. The ladder is the hypotenuse; the wall height is a leg.

Q5. Find the distance between the points (1, 2) and (7, 10).

SOLUTION

$d = \sqrt{(7-1)^2 + (10-2)^2} = \sqrt{36 + 64} = 10$. The distance formula is the Pythagorean theorem with the axes supplying the right angle.

Q6. Find the diagonal of a 12 cm by 5 cm rectangle.

SOLUTION

$d = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$ cm. The 5-12-13 triple, worth recognizing on sight.

Q7. A box measures 3 by 4 by 12. Find the length of its space diagonal.

SOLUTION

Apply the theorem twice, or in one step: $d = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$. The floor diagonal $\sqrt{3^2 + 4^2} = 5$ and the height 12 form a second right triangle: the 5-12-13 triple again.

Q8. An isosceles right triangle has hypotenuse 10. Find its legs.

SOLUTION

With equal legs a : $2a^2 = 100$, so $a = \sqrt{50} = 5\sqrt{2} \approx 7.07$. In every 45-45-90 triangle the hypotenuse is the leg times $\sqrt{2}$.

Q9. Generate Pythagorean triples from $(m, n) = (2, 1)$ and $(3, 2)$ using $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$.

SOLUTION

For $(2, 1)$: $a = 3$, $b = 4$, $c = 5$. For $(3, 2)$: $a = 5$, $b = 12$, $c = 13$. Euclid's formula generates every primitive triple from coprime $m > n$ of opposite parity.

Q10. A triangle has sides 8, 15, 17. Classify its largest angle.

SOLUTION

$8^2 + 15^2 = 64 + 225 = 289 = 17^2$, so the largest angle is exactly 90° . Had $a^2 + b^2$ exceeded c^2 , the angle would be acute; had it fallen short, obtuse. The comparison classifies every triangle.

Answer Key

QUICK ANSWERS

Q1 10 **Q2** 12 **Q3** yes **Q4** 4.8 m **Q5** 10

Q6 13 cm **Q7** 13 **Q8** $5\sqrt{2} \approx 7.07$ **Q9** (3, 4, 5) and (5, 12, 13) **Q10** right angle, since $289 = 17^2$

Revision Sheet

The theorem

In a right triangle, $a^2 + b^2 = c^2$, with c the hypotenuse.

Distance formula

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$; add $(z_2 - z_1)^2$ in 3D.

The converse test

$a^2 + b^2 = c^2$: right. $> c^2$: acute. $< c^2$: obtuse. (c = longest side.)

Special triangles

45-45-90: sides 1 : 1 : $\sqrt{2}$. 30-60-90: sides 1 : $\sqrt{3}$: 2.

Triples worth memorizing

3-4-5, 5-12-13, 7-24-25, 8-15-17, and all their multiples.

Euclid's generator

$a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$ gives a triple for any $m > n$.

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