

COMPLEX NUMBERS

Imaginary Numbers

What i actually is, arithmetic with imaginary numbers, and why they turned out to be indispensable, with a 10-question practice set and full solutions.

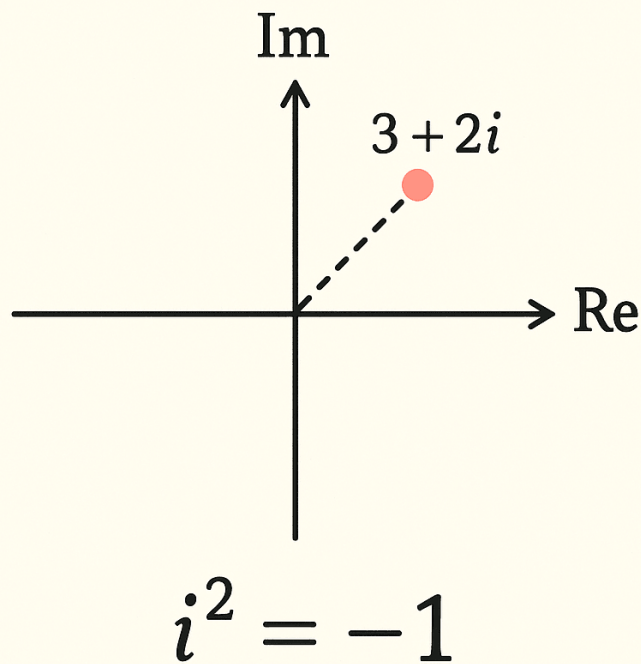
SUBJECT	LEVEL	INCLUDES
Mathematics	High school and early college	Practice set, answer key, revision sheet

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Imaginary numbers extend the real number system by introducing a new unit i defined by $i^2 = -1$. The name 'imaginary' is historical and unfortunate, these numbers are no less real than negative numbers or fractions, and they're absolutely essential in physics, engineering, and pure mathematics. Combined with real numbers, they form the complex numbers, which give algebra and geometry their cleanest unified language.



The complex plane: real axis horizontal, imaginary axis vertical. Every complex number $z = a + bi$ is a point.

The Imaginary Unit

The imaginary unit i is defined by:

$$i^2 = -1 \quad \text{or equivalently} \quad i = \sqrt{-1}$$

This single definition unlocks the square root of any negative number: $\sqrt{-9} = 3i$, $\sqrt{-2} = i\sqrt{2}$, and so on. An imaginary number is anything of the form bi where b is real.

Powers of i

The powers of i cycle through four values:

$$i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \dots$$

To compute i^n for any positive integer, take $n \bmod 4$ and look up the value. Example: i^{2026} : $2026 \bmod 4 = 2$, so $i^{2026} = -1$.

Complex Numbers

A complex number combines a real and imaginary part: $z = a + bi$, where a is the real part and b is the imaginary part. Complex numbers are written with addition and multiplication that follow ordinary algebra plus the rule $i^2 = -1$.

Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$.

Multiplication: $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$.

Complex conjugate: $\overline{a + bi} = a - bi$. Multiplying a number by its conjugate gives a real number: $(a + bi)(a - bi) = a^2 + b^2$.

The Complex Plane

Just as real numbers fit on a number line, complex numbers fit on a 2D plane: the real part on the horizontal axis, the imaginary part on the vertical axis. Each complex number $a + bi$ corresponds to the point (a, b) . The distance from the origin is the **modulus** $|z| = \sqrt{a^2 + b^2}$, and the angle from the positive real axis is the **argument** $\arg(z) = \arctan(b/a)$.

Multiplying complex numbers becomes geometric: their moduli multiply and their arguments add. This is what makes complex numbers the natural language for rotation.

Why They Matter

- **Algebra.** Every non-constant polynomial with complex coefficients has a complex root (Fundamental Theorem of Algebra). The reals don't have this property, $x^2 + 1 = 0$ has no real solution.
- **Electrical engineering.** AC circuits are analyzed using complex impedance $Z = R + jX$ (engineers use j instead of i to avoid clashing with current). Voltage, current, and impedance combine via complex algebra rather than separate amplitude and phase equations.
- **Quantum mechanics.** The wave function ψ is fundamentally complex. The Schrödinger equation has i on the left-hand side; without imaginary numbers, quantum mechanics doesn't exist.
- **Signal processing.** The Fourier transform decomposes signals into complex exponentials $e^{i\omega t}$, a single complex equation that handles amplitude and phase simultaneously.
- **Euler's identity.** $e^{i\pi} + 1 = 0$ connects the five fundamental constants in one line. It's a special case of $e^{i\theta} = \cos \theta + i \sin \theta$, which links exponentials to trigonometry.

Related study notes: Complex Numbers, Quadratic Equations, Euler's Identity, Fourier Series.

Practice Questions

Work each question before reading its solution. The set runs from direct recall and substitution to the applied questions that exams actually use to separate grades.

Q1. Define i , and simplify $\sqrt{-25}$.

SOLUTION

$i = \sqrt{-1}$, so $i^2 = -1$ by definition, a genuinely new mathematical object rather than a real number in disguise. $\sqrt{-25} = \sqrt{25} \times \sqrt{-1} = 5i$.

Q2. Compute i^2 , i^3 , i^4 , and i^5 , and identify the pattern.

SOLUTION

$i^2 = -1$, $i^3 = i^2 \times i = -i$, $i^4 = i^2 \times i^2 = 1$, $i^5 = i^4 \times i = i$. The powers of i cycle with period 4: $i, -1, -i, 1$, repeating forever. Any power of i reduces instantly by finding the remainder when the exponent is divided by 4.

Q3. Find i^{47} using the cyclic pattern.

SOLUTION

$47 = 4 \times 11 + 3$, remainder 3, so $i^{47} = i^3 = -i$. Massive imaginary powers collapse instantly once you know only the remainder mod 4 matters.

Q4. Add $(3 + 4i) + (5 - 2i)$ and multiply $(3 + 4i) \times (5 - 2i)$.

SOLUTION

Addition combines like terms: $(3 + 5) + (4 - 2)i = 8 + 2i$. Multiplication uses FOIL and $i^2 = -1$: $15 - 6i + 20i - 8i^2 = 15 + 14i - 8(-1) = 15 + 14i + 8 = 23 + 14i$. Complex arithmetic follows ordinary algebra rules, with the single extra fact $i^2 = -1$ folded in wherever it appears.

Q5. Simplify $\frac{1}{3 + 2i}$ by rationalizing the denominator.

SOLUTION

Multiply top and bottom by the conjugate $3 - 2i$: $\frac{3 - 2i}{(3 + 2i)(3 - 2i)} = \frac{3 - 2i}{9 + 4} = \frac{3 - 2i}{13} = \frac{3}{13} - \frac{2}{13}i$. Multiplying

by the conjugate always eliminates i from the denominator, because $(a+bi)(a-bi) = a^2 + b^2$, a real number.

Q6. Solve $x^2 + 9 = 0$ over the complex numbers, and explain why this equation has no real solution.

SOLUTION

$x^2 = -9$, so $x = \pm\sqrt{-9} = \pm 3i$. No real number squares to a negative value, since any real number squared is nonnegative; imaginary numbers exist precisely to complete this gap, letting EVERY polynomial equation have a full set of roots.

Q7. Use the quadratic formula to solve $x^2 - 4x + 13 = 0$, and identify the resulting complex conjugate pair.

SOLUTION

$x = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$. Whenever a quadratic's discriminant is negative, the 2 roots always come as a complex conjugate pair, $a + bi$ and $a - bi$, never as unrelated complex numbers.

Q8. State the Fundamental Theorem of Algebra, and explain its connection to imaginary numbers.

SOLUTION

Every polynomial equation of degree n has exactly n roots (counted with multiplicity) over the COMPLEX numbers. Real numbers alone are insufficient, as $x^2 + 1 = 0$ shows; only by extending to complex numbers does every polynomial's root count match its degree exactly, making the complex numbers, not the reals, the natural complete arena for algebra.

Q9. Explain 1 genuine engineering application of complex/imaginary numbers: how electrical engineers use j (their notation for i) to describe AC circuit impedance.

SOLUTION

Resistors, capacitors, and inductors each affect an AC signal differently: resistors resist in-phase, but capacitors and inductors shift the current's phase relative to voltage by 90° in OPPOSITE directions. Representing impedance as a complex number $Z = R + jX$ (with X positive for inductors, negative for capacitors) captures both the magnitude AND the phase shift in a single algebraic object, letting engineers combine circuit elements using ordinary complex-number arithmetic instead of separate trigonometric phase-tracking.

Q10. The Mandelbrot set, a famous fractal, is defined using the complex iteration $z_{n+1} = z_n^2 + c$. Explain in general terms why complex numbers, rather than just real numbers, are essential for this iteration to produce its characteristic 2-dimensional fractal structure.

SOLUTION

A single real number lives on a 1-dimensional number line, so real-number iteration can only ever produce a 1-dimensional pattern. A complex number $a + bi$ inherently carries 2 independent real components (its real and imaginary parts), letting it represent a point in a full 2-dimensional PLANE; iterating in this plane is what allows the Mandelbrot set's famously intricate, infinitely detailed 2-dimensional boundary to emerge, a structure that a purely real-valued iteration could never generate.

Answer Key

QUICK ANSWERS

Q1 $i^2 = -1$; $\sqrt{-25} = 5i$ **Q2** $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i$: cycles every 4 **Q3** $-i$ **Q4** sum: $8 + 2i$; product: $23 + 14i$ **Q5** $\frac{3}{13} - \frac{2}{13}i$
Q6 $x = \pm 3i$ **Q7** $x = 2 \pm 3i$ **Q8** degree- n polynomials always have n complex roots **Q9** impedance $Z = R + jX$ encodes magnitude and 90° phase shift together **Q10** complex numbers encode 2D points, enabling the fractal's 2D structure

Revision Sheet

Definition

$i = \sqrt{-1}$, $i^2 = -1$: a new number, not a disguised real one.

Power cycle

$i, -1, -i, 1$, repeating every 4. Reduce any exponent mod 4.

Arithmetic

Add/subtract like terms; multiply via FOIL using $i^2 = -1$; divide by multiplying by the conjugate.

Conjugates

$(a + bi)(a - bi) = a^2 + b^2$, always real: the tool for rationalizing complex denominators.

Quadratics

Negative discriminant $\Rightarrow$ complex conjugate root pair $a \pm bi$.

Real uses

AC circuit impedance (j -notation), fractals (Mandelbrot set), quantum mechanics, signal processing.

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