ALGEBRA

Function: Notations and Rules

Reading f(x) correctly, evaluating, composing, and transforming functions, and a 10-question practice set with full solutions.

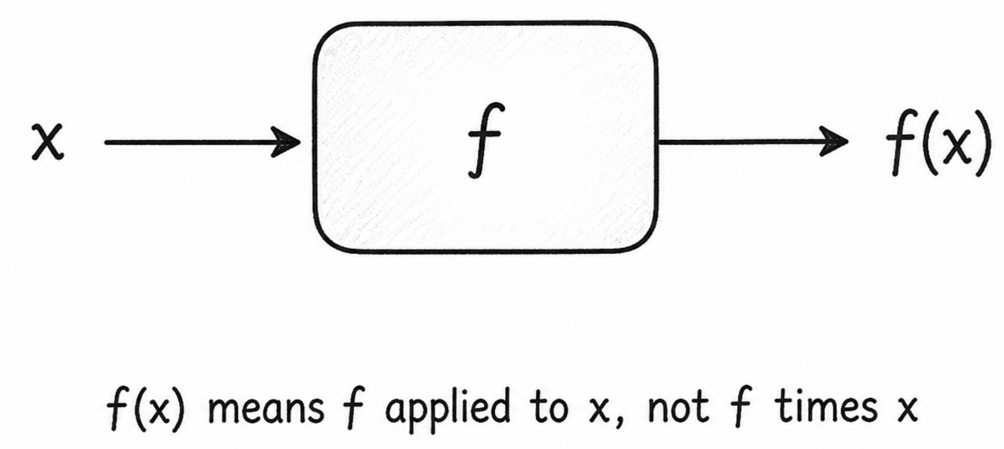
SUBJECT	LEVEL	INCLUDES
Mathematics	High school and early college	Practice set, answer key, revision sheet

Read this note online, with updates: gauravtiwari.org/study-notes/function-notation-rules

INSIDE

What is a Function?

A function in math is a special kind of relation where every input gives you exactly one output. If you pick any element from the domain and run it through the function, you always land on a single, definite value. There is no ambiguity, no branching into two different results.



The function as a machine: one input, one rule, one output. The parentheses mean application, never multiplication.

Formally, f is a function if and only if two conditions hold:

1. Every member of f is an **ordered pair**. That is, $xfy \Rightarrow (x, y) \in f$.
2. If both $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. No input maps to two different outputs.

Here is a quick way to think about it. Consider the set $f = \{(1, 3), (2, 5), (3, 7)\}$. Each first coordinate (1, 2, 3) appears exactly once, so this is a valid function. But if you had $g = \{(1, 3), (1, 5), (2, 7)\}$, the input 1 maps to both 3 and 5. That violates the second condition, so g is not a function.

KEY CONCEPT

The vertical line test is the graphical version of the function definition. If any vertical line crosses a graph more than once, that graph doesn't represent a function. It's the quickest way to check whether a relation qualifies as a function in math.

Notation for Functions

When you write $(x, y) \in f$, you are saying that the ordered pair (x, y) belongs to the function f. Here, x is the **argument** (or input) and y is the **image** (or output). The older notation xfy expresses the same relationship, though you will rarely see it in modern textbooks.

The set of all valid arguments is called the **domain** of the function, and the set of all images (the actual outputs you get) is called the **range**. Keep these two terms sharp in your mind because they come up constantly.

Several equivalent notations exist for expressing $(x, y) \in f$:

- $y = f(x)$ is the standard modern notation you will use most often.
- $y = fx$ is a shorthand sometimes used in algebra and analysis.
- $y : xf$ and $y = x^f$ are older notational forms you may encounter in classical texts.

For example, if $f(x) = 2x + 1$, then $f(3) = 7$. The argument is 3, the image is 7, and you write $(3, 7) \in f$. This notation is fundamental to working with equations across all branches of mathematics.

Domain and Range

The **domain** of a function f is the set of all inputs for which f is defined. The **range** (also called the image) is the set of all outputs that f actually produces. These are not the same as the codomain, which is the larger set that the range sits inside.

Consider $f(x) = x^2$ where $x \in \mathbb{R}$. The domain is all of $\mathbb{R}$, but the range is only $[0, \infty)$ because squaring a real number never gives a negative result. If you are told that $f : \mathbb{R} \to \mathbb{R}$, the codomain is $\mathbb{R}$, but the range is still just the non-negative reals.

Understanding this distinction is critical. When you encounter a function in math, your first question should always be: what is its domain, and what values does it actually take? If you need to find where a function equals zero, that's a separate but related concept covered in the zero of a function study note.

Into Function

A function f is said to be **into** Y if the range of f is a subset of Y, but does not necessarily cover all of Y. In set notation, $R_f \subset Y$. Some elements of Y are left \{}u201cunhit\{}u201d by the function.

For example, let $f : \{1, 2, 3\} \to \{a, b, c, d\}$ be defined by $f(1) = a$, $f(2) = b$, $f(3) = c$. The range is $\{a, b, c\}$, which is a proper subset of the codomain $\{a, b, c, d\}$. The element d has no preimage, so f is an into function. You can think of it as a function that maps \{}u201cinto\{}u201d the codomain without filling it completely.

Onto Function

A function f is **onto** Y (also called **surjective**) if the range of f equals Y exactly. In set notation, $R_f = Y$. Every element in the codomain is hit by at least one element from the domain.

For example, let $g : \{1, 2, 3, 4\} \to \{a, b, c\}$ be defined by $g(1) = a$, $g(2) = b$, $g(3) = c$, $g(4) = a$. The range is $\{a, b, c\}$, which equals the codomain. Every element in the codomain has at least one preimage, so g is onto. Notice that a is hit twice, that is perfectly fine for a surjection.

The general notation for a mapping is $f : X \to Y$, where X is the domain and Y is the codomain. Whether the function is into or onto depends on whether the range fills up all of Y.

One-to-One Function

A function is **one-to-one** (also called **injective**) if it maps distinct elements to distinct elements. No two different inputs ever produce the same output. Formally, f is one-to-one if and only if $x_1 \neq x_2 \Rightarrow f(x_1) \neq$

$f(x_2)$. Equivalently, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Consider $f(x) = 3x + 2$ defined on $\mathbb{R}$. If $f(x_1) = f(x_2)$, then $3x_1 + 2 = 3x_2 + 2$, which gives $x_1 = x_2$. So f is one-to-one. Now consider $g(x) = x^2$ on $\mathbb{R}$. Here $g(2) = 4$ and $g(-2) = 4$, so two different inputs give the same output. This function is not one-to-one.

A function that is both one-to-one and onto is called a **bijection**. Bijections are especially important because they have inverses: you can uniquely reverse the mapping from output back to input.

Restriction of a Function

If you have a function $f : X \to Y$ and you want to limit its domain to a smaller set $A \subseteq X$, you get the **restriction** of f to A. Formally, the restriction is $f \cap (A \times Y)$, which is usually written as $f|_A$. It is still a function from A into Y, but it only \{}u201csees\{}u201d the inputs in A.

For example, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. If you restrict f to $A = [0, \infty)$, you get $f|_{[0,\infty)}(x) = x^2$ for $x \geq 0$. This restricted version is actually one-to-one (since you have removed the negative inputs that caused duplicates), and it is a common technique for making a function invertible.

Extension of a Function

An **extension** goes in the opposite direction. A function f is an extension of a function g if $g \subseteq f$, meaning that f agrees with g everywhere that g is defined, but f may also be defined on additional inputs outside the domain of g.

For example, let $g : \{1, 2\} \to \mathbb{R}$ with $g(1) = 5$ and $g(2) = 8$. Now define $f : \{1, 2, 3\} \to \mathbb{R}$ with $f(1) = 5$, $f(2) = 8$, and $f(3) = 11$. Since f matches g on inputs 1 and 2 and adds a new mapping for 3, we say f is an extension of g. Conversely, g is the restriction of f to $\{1, 2\}$. These two concepts are mirror images of each other.

Operations on Functions

Once you have two functions defined on the same domain, you can combine them using arithmetic operations. If f and g are both functions from X to $\mathbb{R}$, you can define the following:

- **Addition:** $(f + g)(x) = f(x) + g(x)$
- **Subtraction:** $(f - g)(x) = f(x) - g(x)$
- **Multiplication:** $(f \cdot g)(x) = f(x) \cdot g(x)$
- **Division:** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$

The domain of the combined function is the intersection of the domains of f and g. For division, you must also exclude any x where $g(x) = 0$.

TIP

Function operations become much easier when you're comfortable with algebraic manipulation. If you need a refresher on solving and simplifying expressions, check out the equations study note before tackling these problems.

As a concrete example, let $f(x) = x + 1$ and $g(x) = x - 2$, both defined on $\mathbb{R}$. Then $(f + g)(x) = 2x - 1$, $(f \cdot g)(x) = (x + 1)(x - 2) = x^2 - x - 2$, and $\left(\frac{f}{g}\right)(x) = \frac{x+1}{x-2}$ for all $x \neq 2$.

Functions in Trigonometry and Calculus

The concept of a function in math extends far beyond simple algebraic expressions. Trigonometric functions like $\sin(x)$, $\cos(x)$, and $\tan(x)$ are some of the most widely used functions in higher mathematics. Each one maps a real number (an angle) to a specific output value, following the exact same rules covered above.

What makes trigonometric functions interesting is that they're periodic, meaning they repeat their outputs at regular intervals. This periodicity means they aren't one-to-one on their full domain, which is why restrictions (covered earlier) become essential when defining inverse trig functions. For a deeper look at how these functions relate to each other, see the trigonometric identities study note.

In calculus, functions take center stage. Derivatives measure how a function's output changes relative to its input, and integrals measure the accumulation of a function's values over an interval. If you're heading toward calculus, the best calculus books can help you build a strong foundation in function analysis, limits, and continuity.

Key Takeaways

Here are the main ideas you should carry away from this page:

- A function maps each input to exactly one output. The uniqueness of the output is what distinguishes a function from a general relation.
- The standard notation $y = f(x)$ tells you that x is the input and y is the output. The domain is the set of valid inputs, and the range is the set of actual outputs.
- An **into** function does not cover its entire codomain. An **onto** (surjective) function does.
- A **one-to-one** (injective) function never sends two different inputs to the same output. A function that is both one-to-one and onto is a **bijection**.
- **Restriction** shrinks the domain. **Extension** expands it. These are inverse operations and are commonly used to make functions invertible or to define piecewise behavior.
- You can add, subtract, multiply, and divide functions, as long as you respect the domains. For division, always exclude points where the denominator is zero.

Practice Questions

Work each question before reading its solution. The set runs from direct recall and substitution to the applied questions that exams actually use to separate grades.

Q1. For $f(x) = 3x^2 - 2x + 1$, compute $f(2)$ and $f(-1)$.

SOLUTION

$f(2) = 3(4) - 4 + 1 = 9$. $f(-1) = 3(1) + 2 + 1 = 6$. Substitute the input everywhere x appears, keeping the parentheses so signs survive: $(-1)^2 = 1$, not -1.

Q2. Why is $f(x)$ not "f times x"? What operation does the notation actually name?

SOLUTION

f is not a number but a rule; $f(x)$ names the output of applying the rule f to the input x. Reading it as multiplication produces nonsense like canceling the parentheses. The notation is a machine label: input in, output out.

Q3. For $f(x) = x^2 + 1$, expand $f(a + h)$, and explain the most common error.

SOLUTION

$f(a + h) = (a + h)^2 + 1 = a^2 + 2ah + h^2 + 1$. The error is writing $f(a) + f(h)$ or $a^2 + h^2 + 1$: functions do not distribute over addition. The whole input is squared, cross term included; that cross term is exactly what makes derivatives work later.

Q4. For $f(x) = 2x + 3$ and $g(x) = x^2$, compute $f(g(2))$ and $g(f(2))$.

SOLUTION

Inside out: $g(2) = 4$, then $f(4) = 11$. The other order: $f(2) = 7$, then $g(7) = 49$. Composition is not commutative: $f \circ g$ and $g \circ f$ are different functions, and 11 versus 49 proves it on the spot.

Q5. Find the domain of $f(x) = \dfrac{1}{x - 3}$ and $g(x) = \sqrt{x + 2}$.

SOLUTION

f: all reals except 3, where the denominator vanishes. g: $x \geq -2$, so the radicand stays nonnegative.

Domain questions reduce to 2 bans: no division by zero, no even root of a negative.

Q6. For $f(x) = x^2$, describe what $f(x) + 2$, $f(x + 2)$, and $2f(x)$ each do to the graph.

SOLUTION

$f(x) + 2$ shifts up 2. $f(x + 2)$ shifts LEFT 2, opposite to intuition, because inputs reach any given output 2 earlier. $2f(x)$ stretches vertically by 2. Outside the parentheses acts on outputs (vertical); inside acts on inputs (horizontal, reversed).

Q7. Compute the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ for $f(x) = x^2$.

SOLUTION

$\dfrac{(x+h)^2 - x^2}{h} = \dfrac{2xh + h^2}{h} = 2x + h$. As $h \to 0$ this becomes $2x$, the derivative. The difference quotient is the bridge between function notation and calculus, and it works only if $f(x + h)$ was expanded correctly.

Q8. Is $f(x) = x^2$ one-to-one? What does your answer mean for an inverse function?

SOLUTION

No: $f(2) = f(-2) = 4$, two inputs sharing an output, so it fails the horizontal line test. Without one-to-one there is no inverse on the full domain; restricting to $x \geq 0$ repairs it, giving $f^{-1}(x) = \sqrt{x}$. Inverses exist exactly when every output points back to one input.

Q9. For $f(x) = \dfrac{x}{x-1}$, compute $f(f(x))$ and simplify.

SOLUTION

$f(f(x)) = \dfrac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \dfrac{\frac{x}{x-1}}{\frac{x-(x-1)}{x-1}} = \dfrac{x}{1} = x$. This function is its own inverse: applying it twice returns the input. Such involutions exist, and finding one by direct computation is a standard exam surprise.

Q10. A cab charges a flat 50 plus 12 per km. Write the fare as a function, state $C(7)$, and interpret $C(0)$.

SOLUTION

$C(d) = 50 + 12d$. $C(7) = 50 + 84 = 134$. $C(0) = 50$ is the flag-drop: the cost of entering the cab before it moves. Function notation packages a real pricing rule so any question about it becomes an evaluation.

Answer Key

QUICK ANSWERS

Q1 9 and 6 **Q2** application of a rule, not multiplication **Q3** $a^2 + 2ah + h^2 + 1$ **Q4** 11 and 49 **Q5** $x \neq 3$; $x \geq -2$

Q6 up 2; left 2; stretched $\times 2$ **Q7** $2x + h$ **Q8** no; invertible only after restricting the domain **Q9** x: the function is its own inverse **Q10** $C(d) = 50 + 12d$; $C(7) = 134$

Revision Sheet

Reading the notation

$f(x)$ = the rule f applied to input x. Never multiplication.

Domain bans

No division by zero; no even root of a negative. Everything else is allowed unless stated.

Evaluation

Substitute the entire input, parentheses and all: $f(-1)$ and $f(a + h)$ keep their brackets.

Transformations

Outside = vertical as written; inside = horizontal, reversed: $f(x + 2)$ moves left.

Composition

$f(g(x))$: inside first. Order matters; $f \circ g \neq g \circ f$ in general.

The difference quotient

$\frac{f(x+h)-f(x)}{h}$: expand, cancel the h, and calculus begins.

About These Notes

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