NUMBER PATTERNS

Fibonacci Sequence

The recurrence, the golden ratio connection, the useful identities, and a 10-question practice set with full solutions.

SUBJECT	LEVEL	INCLUDES
Mathematics	High school and early college	Practice set, answer key, revision sheet

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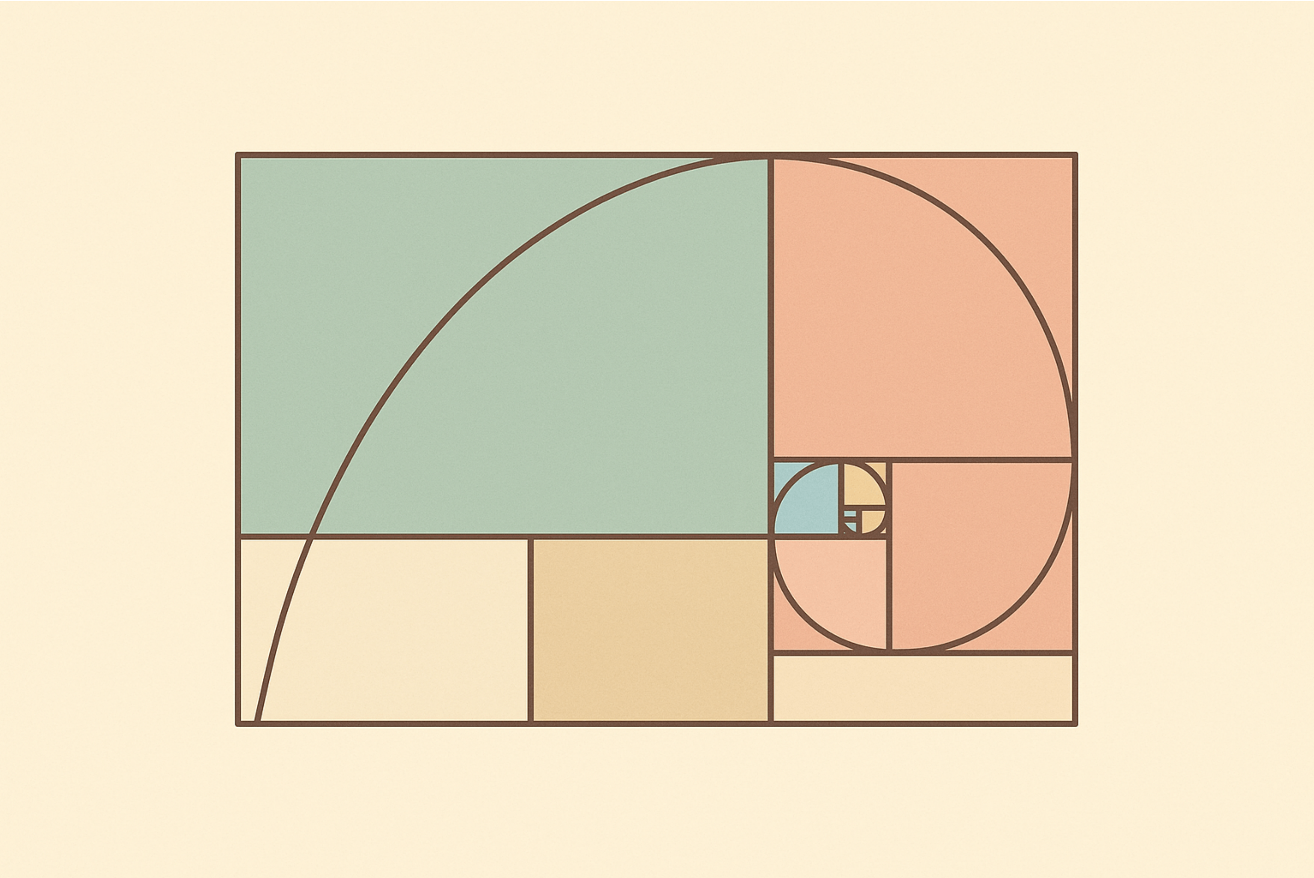
INSIDE

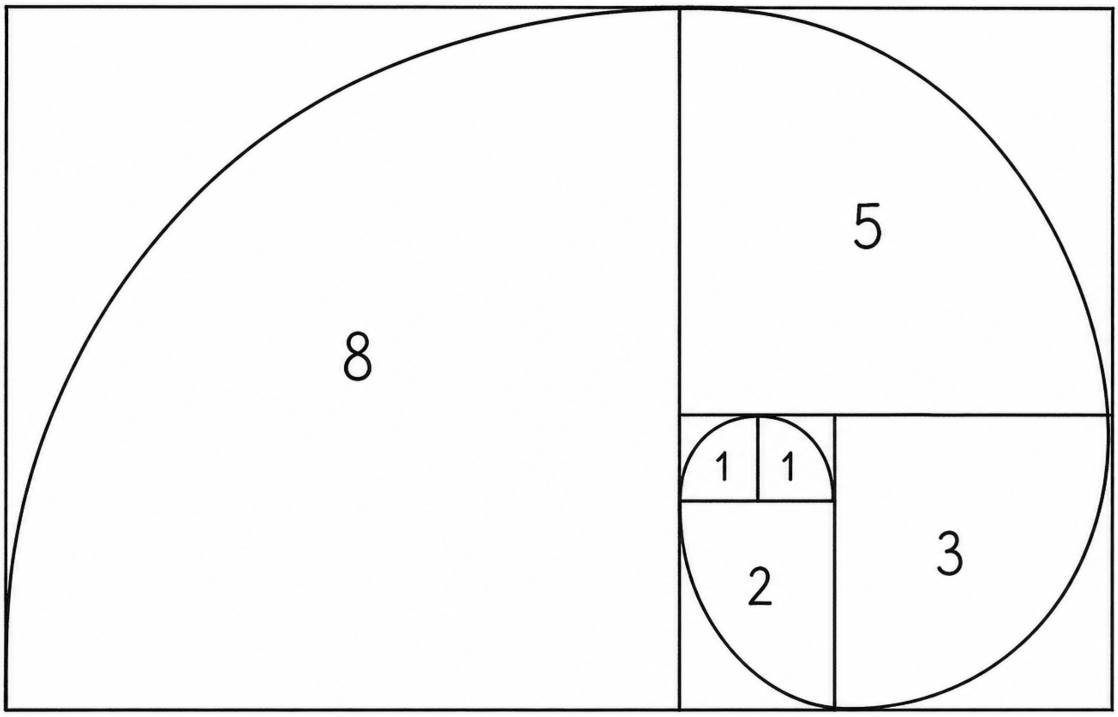
The Fibonacci sequence is the most famous integer sequence in mathematics. Each term is the sum of the two before it, starting from 1 and 1: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, and so on. Named after Leonardo of Pisa (Fibonacci), who introduced it to European mathematics in 1202 through a problem about rabbit breeding, the sequence shows up in unexpected places, sunflower spirals, pinecones, nautilus shells, stock market technical analysis, computer algorithms.



Fibonacci spiral, squares of side 1, 1, 2, 3, 5, 8, 13 arranged in a rectangle traced by a smooth spiral.

The Definition

The Fibonacci sequence is defined by a recurrence relation. The first two terms are seeded; each subsequent term is the sum of the two preceding ones.



The Fibonacci tiling: squares of sides 1, 1, 2, 3, 5, 8 assemble into golden rectangles, and the quarter-circle spiral follows the growth.

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 3$$

Some texts start with $F_0 = 0$ and $F_1 = 1$, which gives 0, 1, 1, 2, 3, 5, 8... The two conventions are equivalent except for the offset. The first 15 terms (using $F_1 = F_2 = 1$):

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

The Original Rabbit Problem

Fibonacci introduced the sequence in his 1202 book Liber Abaci with this puzzle: a pair of rabbits is placed in a field. Each month, every mature pair produces one new pair. New pairs take one month to mature. No rabbits die. How many pairs are there after n months?

Month 1: 1 pair (the original). Month 2: still 1 pair (not yet mature). Month 3: 2 pairs (the original produces). Month 4: 3 pairs (the new pair from month 3 is now mature; original produces again). Month 5: 5 pairs. The sequence is 1, 1, 2, 3, 5, 8, 13, 21, the Fibonacci numbers.

The mathematics behind the puzzle is more important than the puzzle itself. Any process where the next state depends on the sum of two previous states produces a Fibonacci-like growth pattern.

The Golden Ratio Connection

The most beautiful property of the Fibonacci sequence is its connection to the golden ratio $\varphi \approx$ 1.6180339887. The ratio of consecutive Fibonacci numbers approaches φ as n grows large:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180$$

Check it: 2/1 = 2, 3/2 = 1.5, 5/3 ≈ 1.667, 8/5 = 1.6, 13/8 = 1.625, 21/13 ≈ 1.615, 34/21 ≈ 1.619, 55/34 ≈ 1.618. The convergence is fast. By F_{10}/F_9 = 55/34, the ratio is already correct to three decimal places.

This is why the famous Fibonacci spiral, constructed from quarter-circle arcs through nested squares with Fibonacci side lengths, looks visually similar to the true golden spiral. They are not identical, the Fibonacci spiral is built from circular arcs, the golden spiral is a logarithmic spiral with growth factor φ, but the visual difference is small.

The Closed-Form Binet Formula

Despite being defined recursively, the Fibonacci sequence has a closed-form expression discovered by Jacques Binet in 1843 (though it was known earlier to Abraham de Moivre and Daniel Bernoulli):

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \quad \text{where} \quad \varphi = \frac{1+\sqrt{5}}{2}, \;\; \psi = \frac{1-\sqrt{5}}{2}$$

Since $|\psi| < 1$, the term ψ^n shrinks rapidly, and for any practical computation we have $F_n \approx \varphi^n/\sqrt{5}$, rounded to the nearest integer. This is striking: a sequence of integers defined by integer addition has an exact formula involving the irrational $\sqrt{5}$ and the irrational golden ratio.

Identities and Properties

The Fibonacci sequence has dozens of clean identities. A few worth knowing:

- **Sum of first n terms:** $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$
- **Sum of squares:** $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$. Geometrically: the squares tile a Fibonacci rectangle perfectly.
- **Even-indexed sum:** $F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1$
- **Cassini's identity:** $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$. The product of the two neighbors of F_n differs from F_n^2 by exactly ± 1.
- **GCD property:** $\gcd(F_m, F_n) = F_{\gcd(m,n)}$. The greatest common divisor of two Fibonacci numbers is itself a Fibonacci number.

Where the Fibonacci Sequence Shows Up

- **Phyllotaxis.** The arrangement of leaves, seeds, and petals on plants very often follows Fibonacci numbers. Sunflower seed heads have spiral counts that are consecutive Fibonacci numbers (e.g., 34 spirals one way, 55 the other). The reason is that turning by the golden angle (137.5°) between successive leaves maximizes light capture and packing efficiency.
- **Nautilus and snail shells.** These shells grow in approximately logarithmic spirals close to (but not exactly) the golden spiral. The Fibonacci spiral is a piecewise approximation.
- **Computer science.** Fibonacci heaps support fast priority queue operations. The Euclidean algorithm's worst case occurs on consecutive Fibonacci numbers. Many dynamic programming examples use the Fibonacci recurrence as their canonical introduction.
- **Stock market technical analysis.** Fibonacci retracement levels (23.6%, 38.2%, 50%, 61.8%) are widely used by chart-pattern traders. Their actual predictive value is heavily debated, but the convention is entrenched.
- **Music and art composition.** Bartók and Debussy reportedly used Fibonacci-based structures. The 8-bar phrasing common in popular music is a Fibonacci coincidence at best, but it is real that audiences find Fibonacci-proportioned structures pleasing.

What the Fibonacci Sequence Is Not

The Fibonacci sequence is widely mythologized. Three corrections worth making:

- **Sunflowers and pinecones use Fibonacci-related numbers because of the golden angle, not because of mystical reasons.** The golden angle is the rotation that minimizes overlap between successive leaves. Evolution finds it independently in many species.
- **The 'Fibonacci proportions' attributed to ancient buildings (the Parthenon, the Great Pyramid) are almost always coincidence or motivated reasoning.** The dimensions are close to φ if you pick the right measurements; they fail the test if you pick others.
- **The human body is not actually built on Fibonacci proportions.** Da Vinci's Vitruvian Man uses simple integer ratios (1:8 for head to body, for instance), not φ. Modern measurements of finger bones, the navel-to-foot ratio, and similar claims do not reliably match the golden ratio.

Related study notes: Whole Numbers, Set, Complex Numbers.

Practice Questions

Work each question before reading its solution. The set runs from direct recall and substitution to the applied questions that exams actually use to separate grades.

Q1. Write the first 10 Fibonacci numbers, starting from $F_1 = 1, F_2 = 1$.

SOLUTION

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55$. Each term is the sum of the 2 before it: $F_n = F_{n-1} + F_{n-2}$.

Q2. Compute F_{15}.

SOLUTION

Continue the chain: $55, 89, 144, 233, 377, 610$. So $F_{11} = 89, F_{12} = 144, F_{13} = 233, F_{14} = 377, F_{15} = 610$.

Q3. Compute the ratio F_{10}/F_9 and compare it with the golden ratio.

SOLUTION

$F_{10}/F_9 = 55/34 \approx 1.6176$, already close to $\varphi = (1+\sqrt{5})/2 \approx 1.6180$. The ratios of consecutive Fibonacci numbers converge to φ, alternating slightly above and below it.

Q4. Verify the identity $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$ for $n = 6$.

SOLUTION

Left side: $1 + 1 + 2 + 3 + 5 + 8 = 20$. Right side: $F_8 - 1 = 21 - 1 = 20$. The identity holds, and it means partial sums of the sequence are themselves almost Fibonacci numbers.

Q5. Use Binet's formula $F_n = \dfrac{\varphi^n - \psi^n}{\sqrt{5}}$ with $\varphi \approx 1.618, \psi \approx -0.618$ to estimate F_6.

SOLUTION

$\varphi^6 \approx 17.944$ and $\psi^6 \approx 0.056$, so $F_6 \approx (17.944 - 0.056)/2.236 = 17.888/2.236 = 8.0$. Exactly right: $F_6 = 8$. Because $|\psi| < 1$, the ψ^n term shrinks fast, so F_n is just $\varphi^n/\sqrt{5}$ rounded to the nearest integer.

Q6. Every 3rd Fibonacci number is even: $2, 8, 34, 144$. Explain why.

SOLUTION

Track parities. Starting odd, odd, the pattern of sums is odd + odd = even, then odd + even = odd, then even + odd = odd, and the cycle odd, odd, even repeats forever. An even term appears exactly every 3rd position.

Q7. A staircase has 5 steps and you can climb 1 or 2 steps at a time. How many distinct ways can you reach the top?

SOLUTION

Let W_n be the ways to climb n steps. The first move is 1 or 2 steps, so $W_n = W_{n-1} + W_{n-2}$ with $W_1 = 1$, $W_2 = 2$. That gives 1, 2, 3, 5, 8: there are 8 ways. The Fibonacci recurrence appears the moment a count splits by its first move.

Q8. Verify Cassini's identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for $n = 5$.

SOLUTION

$F_6F_4 - F_5^2 = 8 \times 3 - 5^2 = 24 - 25 = -1$, and $(-1)^5 = -1$. It checks. This near-miss between $F_{n+1}F_{n-1}$ and F_n^2 is also the engine behind a classic dissection puzzle that seems to create area from nothing.

Q9. Why do sunflower heads and pinecones so often show consecutive Fibonacci numbers of spirals?

SOLUTION

New florets pack most efficiently when each appears at the golden angle, about 137.5°, from the previous one, because φ is in a precise sense the number hardest to approximate by fractions. Spiral counts made by such packing come out as consecutive Fibonacci numbers. It is efficient geometry, not numerology.

Q10. Using $F_n \approx \varphi^n/\sqrt{5}$, estimate F_{20}.

SOLUTION

$\varphi^{20} \approx 15{,}127$, so $F_{20} \approx 15{,}127/2.236 \approx 6{,}765$. The exact value is 6,765. The approximation is already exact after rounding, which shows how completely φ controls the sequence's growth.

Answer Key

QUICK ANSWERS

Q1 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 **Q2** 610 **Q3** $55/34 \approx 1.6176 \approx \varphi$ **Q4** both sides equal 20 **Q5** $F_6 = 8$

Q6 parity cycles odd, odd, even **Q7** 8 ways **Q8** $24 - 25 = -1 = (-1)^5$ **Q9** golden-angle packing forces Fibonacci spiral counts **Q10** 6,765

Revision Sheet

The recurrence

$F_n = F_{n-1} + F_{n-2}$, $F_1 = F_2 = 1$. Sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

Sum identity

$F_1 + \cdots + F_n = F_{n+2} - 1$

Golden ratio

$F_{n+1}/F_n \to \varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$

Cassini

$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$

Binet's formula

$F_n = \dfrac{\varphi^n - \psi^n}{\sqrt{5}}$; in practice $F_n = \mathrm{round}(\varphi^n/\sqrt{5})$.

Counting appearances

Any count that splits by its first move (steps, tilings, rabbit pairs) obeys the Fibonacci recurrence.

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