COMPLEX ANALYSIS

Euler's Identity

The equation linking 5 fundamental constants, why it works, and where it shows up in engineering, with a 10-question practice set and full solutions.

SUBJECT	LEVEL	INCLUDES
Mathematics	High school and early college	Practice set, answer key, revision sheet

Read this note online, with updates: gauravtiwari.org/study-notes/euler-identity

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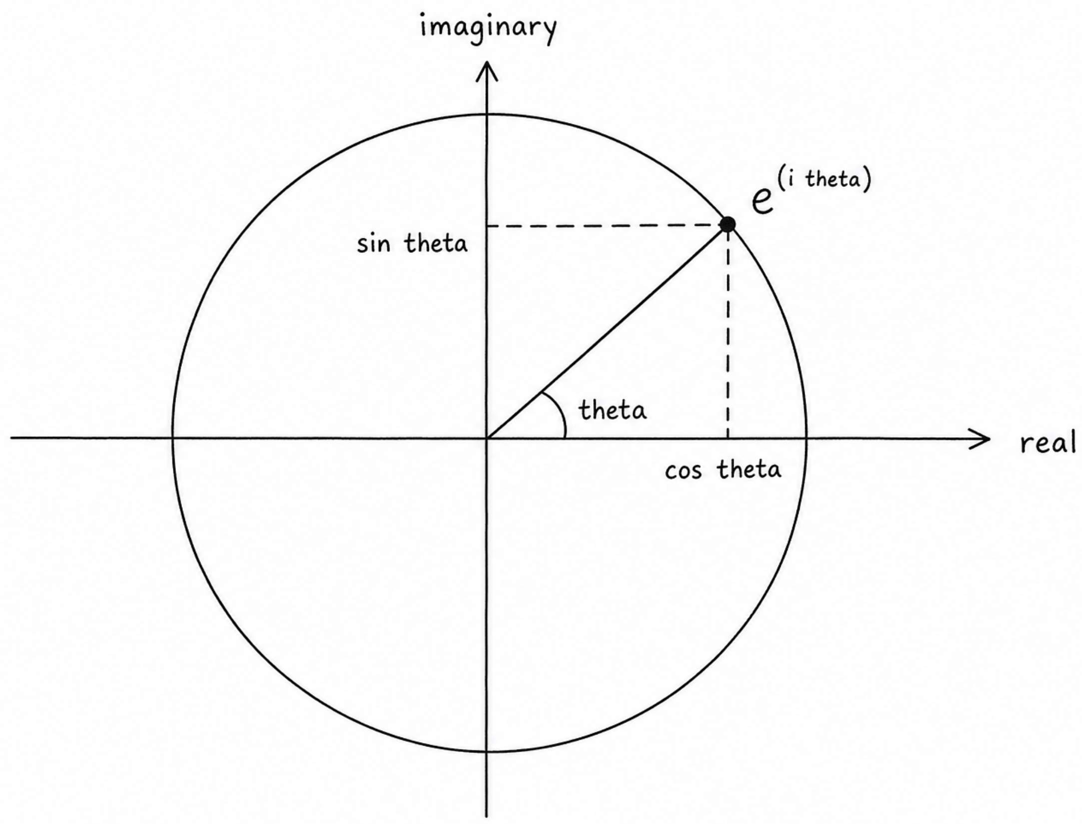
Euler's identity, $e^{i\pi}+1 = 0$, is widely considered the most beautiful equation in mathematics. In one line it links the five most fundamental mathematical constants, 0, 1, i, π, and e, using the three basic operations of addition, multiplication, and exponentiation, exactly once each, with no extraneous symbols. It's a special case of Euler's broader formula $e^{i\theta} = \cos\theta + i\sin\theta$, which connects exponential functions to trigonometry through the complex plane.



Euler's identity: e^(iπ) + 1 = 0. Often called the most beautiful equation in mathematics.

The Identity

Euler's identity:

$$e^{i\pi} + 1 = 0$$

The five constants in one equation:

- **0**: the additive identity.

- **1**: the multiplicative identity.
- π: the ratio of a circle's circumference to its diameter, ≈ 3.14159.
- e: the base of the natural logarithm, ≈ 2.71828.
- i: the imaginary unit, $\sqrt{-1}$.

Each appears exactly once. Three operations, addition, multiplication, exponentiation, also each appear exactly once.

Euler's Formula

The identity follows immediately from Euler's broader **formula**:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

valid for all real θ. Plugging in $\theta = \pi$: $e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0 = -1$. Rearranging: $e^{i\pi} + 1 = 0$. Done.

Why Euler's Formula Is True

Three independent derivations, each illuminating:

(**1**) **Taylor series.** Expand each side:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Substitute $x = i\theta$ and use $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, ... Group real and imaginary terms. The real part matches the Taylor series for $\cos\theta$, and the imaginary part matches $\sin\theta$. Done.

(**2**) **Differential equations.** Both sides satisfy $y' = iy$ with $y(0) = 1$. By uniqueness of solutions, they must be equal.

(**3**) **Complex plane geometry.** $e^{i\theta}$ represents the point on the unit circle at angle θ from the positive real axis. Multiplying by $e^{i\theta}$ rotates the complex plane by θ. The point at angle π on the unit circle is exactly -1.

Consequences and Applications

- **Trigonometric identities.** All standard identities (sum-to-product, double-angle, half-angle) follow from $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ by expanding both sides.
- **Complex exponentials in physics.** Wave equations of every kind are written using $e^{i\omega t}$, quantum mechanics, electromagnetism, circuit theory, signal processing, acoustics, optics.
- **Fourier transforms.** The Fourier transform decomposes a function into a sum of $e^{i\omega t}$ terms. All of signal processing rests on this.
- **Roots of unity.** The n solutions of $z^n = 1$ are $e^{2\pi ik/n}$ for $k = 0, 1, \ldots, n-1$, equally spaced points on the unit circle. They're used in everything from cyclic codes to discrete Fourier transforms.
- **Electrical engineering.** Complex impedance $Z = R + iX$ lets phasor algebra replace messy sinusoidal time-domain calculations.

Why Mathematicians Find It Beautiful

Three reasons combine. *Economy:* five fundamental constants and three operations, each appearing exactly once. *Surprise:* e is connected to compound growth, π to circles, i to algebra, they have no obvious relationship, yet they combine into a clean integer identity. *Depth:* the identity isn't a coincidence; it falls out of the deep connection between exponential functions and rotation in the complex plane. Polls of mathematicians regularly rank it as the most beautiful equation in mathematics.

Related study notes: Imaginary Numbers, Exponential Function, Trigonometry, Fourier Series.

Practice Questions

Work each question before reading its solution. The set runs from direct recall and substitution to the applied questions that exams actually use to separate grades.

Q1. Write Euler's identity and name the 5 constants it connects.

SOLUTION

$e^{i\pi} + 1 = 0$. It connects e (growth), i (the imaginary unit), π (geometry), 1 (multiplicative identity), and 0 (additive identity), 1 equation from 5 seemingly unrelated corners of mathematics, often called the most beautiful equation for exactly that reason.

Q2. State Euler's formula, the more general result Euler's identity comes from.

SOLUTION

$e^{i\theta} = \cos\theta + i\sin\theta$, for any real θ. This links complex exponentials to trigonometry directly: raising e to an imaginary power does not diverge or misbehave, it traces a point on the unit circle.

Q3. Derive Euler's identity from Euler's formula by substituting $\theta = \pi$.

SOLUTION

$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1$. Rearranged: $e^{i\pi} + 1 = 0$. The identity is simply the formula evaluated at 1 specific, geometrically clean angle: halfway around the unit circle.

Q4. Interpret $e^{i\theta}$ geometrically. What curve does it trace as θ runs from 0 to 2π?

SOLUTION

It traces the unit circle in the complex plane, starting at $(1, 0)$ when $\theta = 0$ and sweeping counterclockwise. Multiplying by $e^{i\theta}$ is literally a ROTATION by angle θ; this is why complex exponentials are the natural language for anything involving rotation or periodic motion.

Q5. Use Euler's formula to derive $\cos\theta$ and $\sin\theta$ in terms of complex exponentials.

SOLUTION

From $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$: adding gives $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, subtracting gives

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$. These identities convert trigonometric integrals and derivatives into simpler exponential algebra, a standard engineering shortcut.

Q6. Evaluate $e^{i\pi/2}$ and $e^{2\pi i}$, and interpret both results geometrically.

SOLUTION

$e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$: a quarter-turn lands exactly on the imaginary axis. $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$: a full turn returns to the start, confirming $e^{i\theta}$ is periodic with period 2π, exactly matching the circle it traces.

Q7. Multiply 2 complex numbers in polar exponential form: $(3e^{i\pi/4}) \times (2e^{i\pi/3})$. What does this reveal about complex multiplication?

SOLUTION

Magnitudes multiply, angles add: $3 \times 2 = 6$, $\pi/4 + \pi/3 = 7\pi/12$, giving $6e^{i7\pi/12}$. This is the entire reason exponential form exists: multiplying complex numbers, geometrically confusing in $a + bi$ form, becomes simple addition of angles and multiplication of lengths in exponential form.

Q8. Electrical engineers write an AC voltage as $V(t) = V_0 e^{i\omega t}$ instead of $V_0\cos(\omega t)$. Explain the practical advantage.

SOLUTION

Differentiating or integrating $e^{i\omega t}$ just multiplies by $i\omega$ or divides by it, a plain algebraic operation, while differentiating $\cos(\omega t)$ forces a trigonometric identity ($\cos \to -\sin$) at every step. Circuit analysis with resistors, inductors, and capacitors becomes ordinary complex-number algebra (impedances) instead of a swamp of trig identities; only at the final step does an engineer take the real part to recover the physical voltage.

Q9. A quantum wavefunction includes a phase factor $e^{i\phi}$. Why does this factor never affect any measurable probability, and what does this reveal about the structure of quantum mechanics?

SOLUTION

Probability comes from $|\psi|^2$, and $|e^{i\phi}|^2 = 1$ for any real ϕ, since $e^{i\phi}$ always sits on the unit circle. Multiplying a wavefunction by an overall phase changes nothing observable: it reveals that quantum states are, strictly, equivalence classes under global phase, not literal complex-valued objects, which is a foundational and often underappreciated structural fact about the theory.

Q10. Solve $x^4 = 1$ over the complex numbers using Euler's formula, finding all 4 roots.

SOLUTION

Write $1 = e^{i\cdot 0}$, then also $e^{i\cdot 2\pi k}$ for any integer k. Fourth roots: $x = e^{i\cdot 2\pi k/4}$ for $k = 0, 1, 2, 3$, giving $1, i, -1, -i$, evenly spaced around the unit circle every 90°. Complex exponential form is the standard method for finding ALL n-th roots of any number, real solving methods routinely miss the complex ones entirely.

Answer Key

QUICK ANSWERS

Q1 $e, i, \pi, 1, 0$ **Q2** $e^{i\theta} = \cos\theta + i\sin\theta$ **Q3** $\cos\pi + i\sin\pi = -1$, so $e^{i\pi} + 1 = 0$ **Q4** the unit circle, traced counterclockwise **Q5** $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

Q6 $e^{i\pi/2} = i$; $e^{2\pi i} = 1$ **Q7** $6e^{i7\pi/12}$: magnitudes multiply, angles add **Q8** derivatives/integrals become simple multiplication by $i\omega$ **Q9** $|e^{i\phi}| = 1$ always; global phase is unobservable **Q10** $1, i, -1, -i$

Revision Sheet

The identity

$e^{i\pi} + 1 = 0$: connects e, i, π, 1, 0 in one equation.

Multiplication rule

Polar/exponential form: magnitudes multiply, angles add. The reason engineers use it.

Euler's formula

$e^{i\theta} = \cos\theta + i\sin\theta$: traces the unit circle as θ sweeps.

Applications

AC circuit impedance, quantum phase factors, signal processing (Fourier transforms).

Trig from exponentials

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

Roots of unity

n-th roots of 1: $e^{i\cdot 2\pi k/n}$ for $k = 0, \ldots, n-1$, evenly spaced on the unit circle.

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