ALGEBRA

# Binomial Theorem

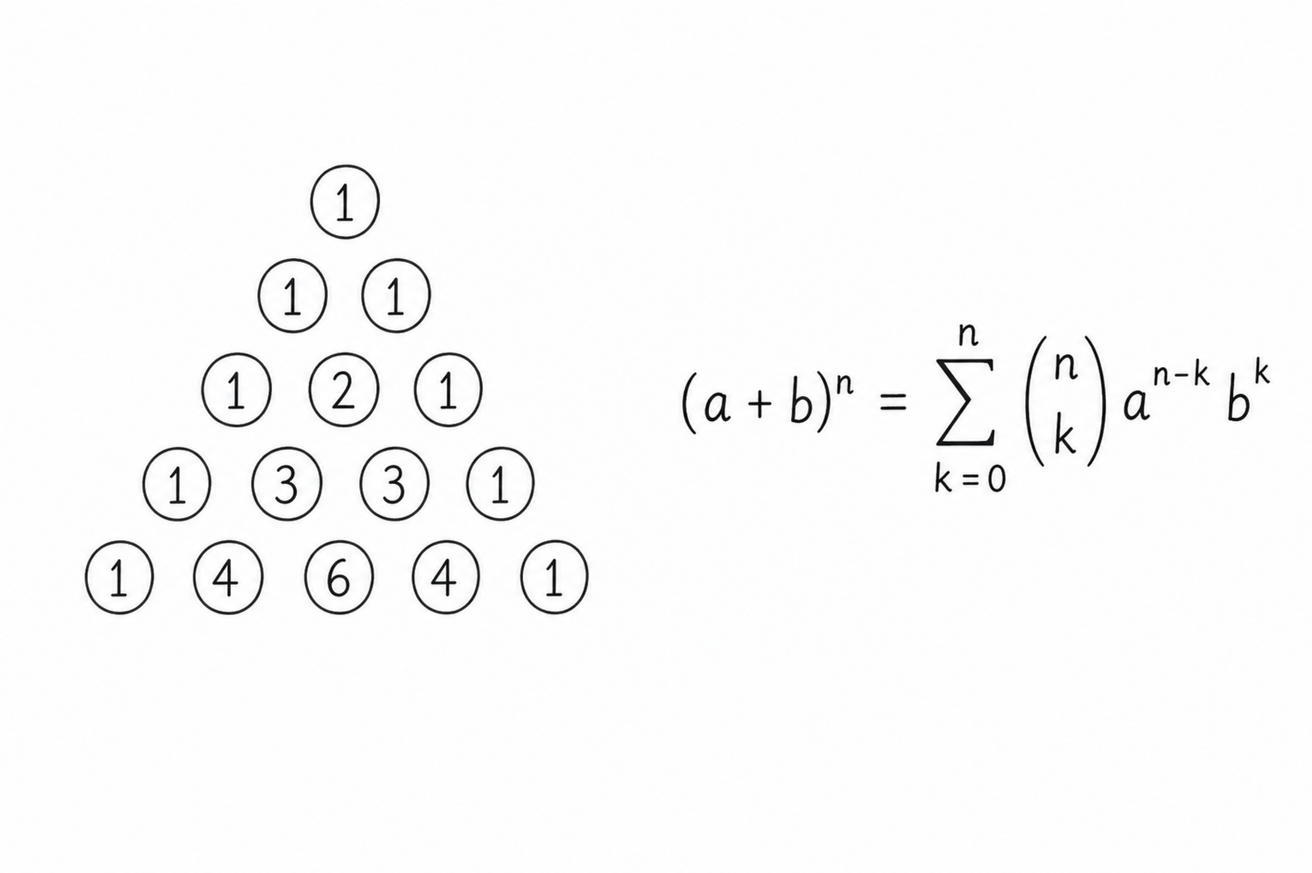
Expanding powers without multiplying, Pascal's triangle, finding specific terms, and a 10-question practice set with full solutions.

| SUBJECT | LEVEL | INCLUDES |
|---|---|---|
| Mathematics | High school and early college | Practice set, answer key, revision sheet |

Read this note online, with updates: gauravtiwari.org/study-notes/binomial-theorem

INSIDE

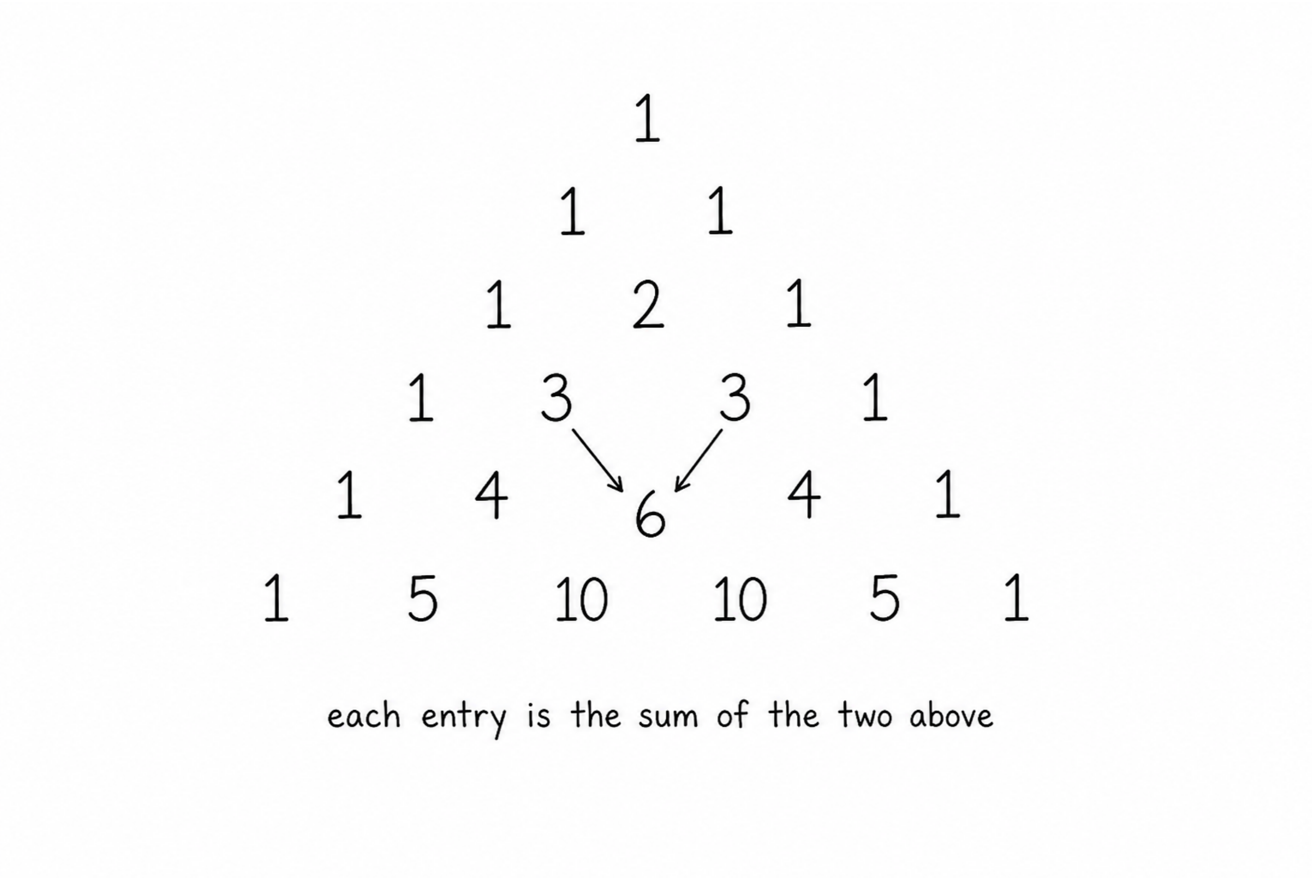
The binomial theorem gives a clean formula for expanding any power of a sum: $(a + b)^n$ for any non-negative integer $n$. It tells you exactly what coefficients appear in front of each term, without having to multiply the expression out manually. The coefficients follow Pascal's triangle, the same triangle of numbers that shows up in combinatorics, probability, and surprisingly often in the rest of mathematics. Isaac Newton extended the theorem to non-integer exponents in 1665, but the integer case has been known since the medieval Arab and Indian mathematicians.




Pascal's triangle and the binomial theorem.

## The Theorem

For any positive integer $n$:




Pascal's triangle: each entry is the sum of the 2 above it, and row n gives the coefficients of $(x + y)^n$.

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

where the binomial coefficient $\binom{n}{k}$ (read 'n choose k') is:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

This single formula gives you every term in the expansion of $(a + b)^n$ without manual multiplication.

## Worked Examples

**$(a + b)^2$:** coefficients $\binom{2}{0}, \binom{2}{1}, \binom{2}{2} = 1, 2, 1$. Expansion: $a^2 + 2ab + b^2$. The familiar square-of-a-sum identity falls out directly.

**$(a + b)^3$:** coefficients $1, 3, 3, 1$. Expansion: $a^3 + 3a^2b + 3ab^2 + b^3$.

**$(a + b)^4$:** coefficients $1, 4, 6, 4, 1$. Expansion: $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

$(2x+3)^5$: coefficients 1, 5, 10, 10, 5, 1. Substituting $a = 2x, b = 3$: $(2x)^5 + 5(2x)^4(3) + 10(2x)^3(3)^2 + 10(2x)^2(3)^3 + 5(2x)(3)^4 + (3)^5 = 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$.

## Pascal's Triangle

The binomial coefficients form Pascal's triangle, each row gives the coefficients of the corresponding binomial expansion.

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \end{array}$$

Each entry is the sum of the two directly above it. The triangle was known in China (Yang Hui, 13th century), Persia (al-Karaji, 10th century), and Europe (Pascal, 1654). The full structure is older than Pascal's claim.

## Why It Works

When you multiply out $(a+b)^n = (a+b)(a+b)\cdots(a+b)$ with $n$ factors, each term in the expansion comes from choosing either $a$ or $b$ from each factor. A term with $k$ copies of $b$ and $n-k$ copies of $a$ is $a^{n-k}b^k$. The number of ways to choose which $k$ of the $n$ factors contribute a $b$ is exactly $\binom{n}{k}$. That's the binomial coefficient. So the term $a^{n-k}b^k$ appears $\binom{n}{k}$ times in the expansion.

# Applications

- **Combinatorics.** $\binom{n}{k}$ is the number of ways to choose $k$ items from $n$, used everywhere in counting problems.
- **Probability.** The binomial distribution gives the probability of getting exactly $k$ successes in $n$ independent trials, each with success probability $p$: $P(X = k) = \binom{n}{k}p^k(1 - p)^{n-k}$.
- **Newton's generalized binomial.** Newton extended the theorem to fractional and negative exponents using infinite series: $(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$ for $|x| < 1$. This is the basis of Taylor approximations for fractional powers.
- **Approximation.** The first two terms of $(1 + x)^n$, namely $1 + nx$, give the linear approximation widely used in physics for small $x$.

Related study notes: Permutations and Combinations, Taylor Series, Normal Distribution, Exponential Function.

# Practice Questions

Work each question before reading its solution. The set runs from direct recall and substitution to the applied questions that exams actually use to separate grades.

**Q1.** State the binomial theorem and expand $(x+y)^3$.

SOLUTION

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

For $n = 3$: $x^3 + 3x^2y + 3xy^2 + y^3$, coefficients 1, 3, 3, 1 from Pascal's row 3. Each term's powers sum to $n$: a built-in error check.

**Q2.** Why is the coefficient of $x^{n-k}y^k$ exactly $\binom{n}{k}$? Give the counting argument.

SOLUTION

Expanding $(x+y)^n$ multiplies $n$ brackets; each term picks $x$ or $y$ from every bracket. A term with $k$ y's arises once for each way of CHOOSING which $k$ brackets contribute $y$: that count is $\binom{n}{k}$. The theorem is combinatorics wearing algebra's clothes.

**Q3.** Expand $(2x-3)^4$.

SOLUTION

With $a = 2x$, $b = -3$, coefficients 1, 4, 6, 4, 1: $16x^4 - 96x^3 + 216x^2 - 216x + 81$. The alternating signs come from odd powers of $-3$; losing a sign or forgetting to raise the 2 along with $x$ are the standard slips.

**Q4.** Find the coefficient of $x^5$ in $(1+x)^{12}$.

SOLUTION

$\binom{12}{5} = 792$. One term, no expansion: the theorem's general term formula exists precisely so you never expand 13 terms to read off one.

**Q5.** Find the term independent of $x$ in $\left(x^2 + \frac{1}{x}\right)^9$.

SOLUTION

General term: $\binom{9}{k}(x^2)^{9-k}x^{-k} = \binom{9}{k}x^{18-3k}$. Zero exponent: $k = 6$, giving $\binom{9}{6} = 84$. Set the exponent to the target and solve for $k$: the routine works for any requested power.

**Q6.** Use the theorem to estimate $(1.02)^{10}$ to 3 decimal places.

SOLUTION

$(1 + 0.02)^{10} = 1 + 10(0.02) + 45(0.0004) + 120(0.000008) + \cdots = 1 + 0.2 + 0.018 + 0.00096 \approx 1.219$. Three terms deliver bank-grade accuracy because higher powers of 0.02 collapse; this truncation is the engine behind quick compound-interest and error estimates.

**Q7.** Prove that the coefficients of any row sum to $2^n$, and interpret the identity.

SOLUTION

Set $x = y = 1$: $(1 + 1)^n = \sum \binom{n}{k} = 2^n$. Interpretation: both sides count subsets of an $n$-element set, by size on the left, in total on the right. Substituting clever values into the theorem is a proof factory: $x = 1, y = -1$ similarly shows alternating coefficients cancel.

**Q8.** Find the middle term of $(x - 2y)^8$.

SOLUTION

Nine terms, so the middle is the 5th: $k = 4$: $\binom{8}{4}x^4(-2y)^4 = 70 \times 16\, x^4y^4 = 1120x^4y^4$. Even $n$ gives one middle term at $k = n/2$; odd $n$ gives two.

**Q9.** In the expansion of $(1 + x)^n$, consecutive coefficients are in ratio 7 : 42 : 105 at some position. Wait: simplify to find $n$ given $\binom{n}{1} : \binom{n}{2} = 1 : 6$.

SOLUTION

$\frac{\binom{n}{2}}{\binom{n}{1}} = \frac{n-1}{2} = 6$, so $n = 13$. Ratios of consecutive binomial coefficients collapse to $\frac{n-k}{k+1}$, which converts ratio conditions into linear equations. That collapse is the tool for every such puzzle.

**Q10.** A fair coin is tossed 10 times. Use a binomial coefficient to count sequences with exactly 4 heads, and give the probability.

SOLUTION

Sequences: $\binom{10}{4} = 210$; each of the $2^{10} = 1024$ sequences is equally likely, so $P = 210/1024 \approx 0.205$. The binomial DISTRIBUTION in statistics is the binomial THEOREM with probabilities substituted for $x$ and $y$: $(p + q)^n$ expanded term by term.

# Answer Key

QUICK ANSWERS

**Q1** $x^3 + 3x^2y + 3xy^2 + y^3$ **Q2** choose which $k$ brackets give $y$ **Q3** $16x^4 - 96x^3 + 216x^2 - 216x + 81$

**Q4** 792 **Q5** 84

**Q6** $\approx 1.219$ **Q7** set $x = y = 1$: subsets counted 2 ways **Q8** $1120x^4y^4$ **Q9** $n = 13$ **Q10** 210 ways; $\approx 20.5\%$

# Revision Sheet

### The theorem

$(x+y)^n = \sum_k \binom{n}{k}x^{n-k}y^k$; powers in each term sum to $n$.

### Classic identities

Row sums $2^n$ (set $x=y=1$); alternating sum 0 (set $y = -1$).

### Coefficients

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$: Pascal's triangle rows; symmetric $\binom{n}{k} = \binom{n}{n-k}$.

### Approximation

$(1+\varepsilon)^n \approx 1 + n\varepsilon + \binom{n}{2}\varepsilon^2$ for small $\varepsilon$.

### General term

$T_{k+1} = \binom{n}{k}a^{n-k}b^k$: set the exponent you need, solve for $k$.

### To statistics

$(p+q)^n$ term $\binom{n}{k}p^kq^{n-k}$ = probability of $k$ successes.

# About These Notes

This note is part of a free study-notes series on gauravtiwari.org covering mathematics, physics, chemistry, and biology: the concepts that keep deciding grades in high school, board, and entrance exams.

| | | | |
|---|---|---|---|
| **2008** | **2,000+** | **125+** | **M.Sc.** |
| Writing and teaching online since | Articles published on gauravtiwari.org | Free study notes in this series | Mathematics, with physics and chemistry training |

Gaurav Tiwari is a developer, educator, and founder of Gatilab. The study-notes series exists because most exam prep material is either a full textbook or a thin definition page, and revision needs something in between: one concept, complete, with the traps named and the practice attached.

GET THE FULL SERIES

Every note in the series is free to read online and free to download as a PDF like this one. Browse the complete collection at **gauravtiwari.org/free-study-notes-exam-prep** and pick the topics your next exam actually covers.



Gaurav Tiwari

gauravtiwari.org