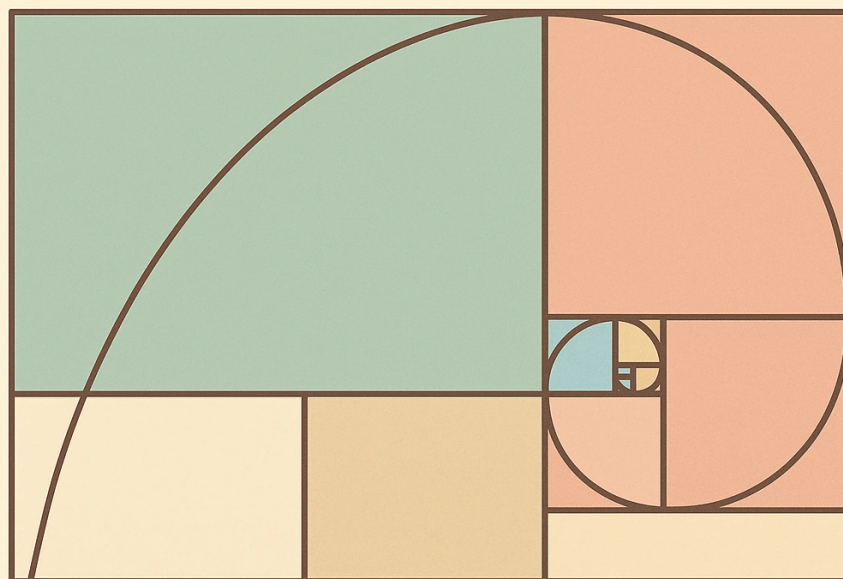


# Fibonacci Sequence

[Read this note online](#)

The Fibonacci sequence is the most famous integer sequence in mathematics. Each term is the sum of the two before it, starting from 1 and 1: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, and so on. Named after Leonardo of Pisa (Fibonacci), who introduced it to European mathematics in 1202 through a problem about rabbit breeding, the sequence shows up in unexpected places, sunflower spirals, pinecones, nautilus shells, stock market technical analysis, computer algorithms.

A Fibonacci sequence adds its previous 2 terms. The indexing convention matters when you identify a numbered term.



Fibonacci spiral, squares of side 1, 1, 2, 3, 5, 8, 13 arranged in a rectangle traced by a smooth spiral.

## The Definition

The Fibonacci sequence is defined by a recurrence relation. The first two terms are seeded; each subsequent term is the sum of the two preceding ones.

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 3$$

Some texts start with  $F_0 = 0$  and  $F_1 = 1$ , which gives 0, 1, 1, 2, 3, 5, 8... The two conventions are equivalent except for the offset. The first 15 terms (using  $F_1 = F_2 = 1$ ):

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \dots$$

## The Original Rabbit Problem

Fibonacci introduced the sequence in his 1202 book Liber Abaci with this puzzle: a pair of rabbits is placed in a field. Each month, every mature pair produces one new pair. New pairs take one month to mature. No rabbits die. How many pairs are there after  $n$  months?

Month 1: 1 pair (the original). Month 2: still 1 pair (not yet mature). Month 3: 2 pairs (the original produces). Month 4: 3 pairs (the new pair from month 3 is now mature; original produces again). Month 5: 5 pairs. The sequence is 1, 1, 2, 3, 5, 8, 13, 21, the Fibonacci numbers.

The mathematics behind the puzzle is more important than the puzzle itself. Any process where the next state depends on the sum of two previous states produces a Fibonacci-like growth pattern.

## The Golden Ratio Connection

The most beautiful property of the Fibonacci sequence is its connection to the **golden ratio**  $\varphi \approx 1.6180339887$ . The ratio of consecutive Fibonacci numbers approaches  $\varphi$  as  $n$  grows large:

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \varphi = \frac{1 + \sqrt{5}}{2} \approx 1.6180$$

Check it:  $2/1 = 2$ ,  $3/2 = 1.5$ ,  $5/3 \approx 1.667$ ,  $8/5 = 1.6$ ,  $13/8 = 1.625$ ,  $21/13 \approx 1.615$ ,  $34/21 \approx 1.619$ ,  $55/34 \approx 1.618$ . The convergence is fast. By  $F_{10}/F_9 = 55/34$ , the ratio is already correct to three decimal places.

This is why the famous Fibonacci spiral, constructed from quarter-circle arcs through nested squares with Fibonacci side lengths, looks visually similar to the true golden spiral. They are not identical, the Fibonacci spiral is built from circular arcs, the golden spiral is a logarithmic spiral with growth factor  $\varphi$ , but the visual difference is small.

# The Closed-Form Binet Formula

Despite being defined recursively, the Fibonacci sequence has a closed-form expression discovered by Jacques Binet in 1843 (though it was known earlier to Abraham de Moivre and Daniel Bernoulli):

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \quad \text{where} \quad \varphi = \frac{1 + \sqrt{5}}{2}, \quad \psi = \frac{1 - \sqrt{5}}{2}$$

Since  $|\psi| < 1$ , the term  $\psi^n$  shrinks rapidly, and for any practical computation we have  $F_n \approx \varphi^n / \sqrt{5}$ , rounded to the nearest integer. This is striking: a sequence of integers defined by integer addition has an exact formula involving the irrational  $\sqrt{5}$  and the irrational golden ratio.

## Identities and Properties

The Fibonacci sequence has dozens of clean identities. A few worth knowing:

- **Sum of first n terms:**  $F_1 + F_2 + \dots + F_n = F_{n+2} - 1$
- **Sum of squares:**  $F_1^2 + F_2^2 + \dots + F_n^2 = F_n F_{n+1}$ . Geometrically: the squares tile a Fibonacci rectangle perfectly.
- **Even-indexed sum:**  $F_2 + F_4 + F_6 + \dots + F_{2n} = F_{2n+1} - 1$
- **Cassini's identity:**  $F_{n-1} F_{n+1} - F_n^2 = (-1)^n$ . The product of the two neighbors of  $F_n$  differs from  $F_n^2$  by exactly  $\pm 1$ .
- **GCD property:**  $\gcd(F_m, F_n) = F_{\gcd(m, n)}$ . The greatest common divisor of two Fibonacci numbers is itself a Fibonacci number.

## Where the Fibonacci Sequence Shows Up

- **Phyllotaxis.** The arrangement of leaves, seeds, and petals on plants very often follows Fibonacci numbers. Sunflower seed heads have spiral counts that are consecutive Fibonacci numbers (e.g., 34 spirals one way, 55 the other). The reason is that turning by the golden angle ( $137.5^\circ$ ) between successive leaves maximizes light capture and packing efficiency.
- **Nautilus and snail shells.** These shells grow in approximately logarithmic spirals close to (but not exactly) the golden spiral. The Fibonacci spiral is a piecewise approximation.
- **Computer science.** Fibonacci heaps support fast priority queue operations. The Euclidean algorithm's worst case occurs on consecutive Fibonacci numbers. Many dynamic programming examples use the Fibonacci recurrence as their canonical introduction.
- **Stock market technical analysis.** Fibonacci retracement levels (23.6%, 38.2%, 50%, 61.8%) are

widely used by chart-pattern traders. Their actual predictive value is heavily debated, but the convention is entrenched.

- **Music and art composition.** Bartók and Debussy reportedly used Fibonacci-based structures. The 8-bar phrasing common in popular music is a Fibonacci coincidence at best, but it is real that audiences find Fibonacci-proportioned structures pleasing.

## What the Fibonacci Sequence Is Not

The Fibonacci sequence is widely mythologized. Three corrections worth making:

- **Sunflowers and pinecones use Fibonacci-related numbers because of the golden angle, not because of mystical reasons.** The golden angle is the rotation that minimizes overlap between successive leaves. Evolution finds it independently in many species.
- **The 'Fibonacci proportions' attributed to ancient buildings (the Parthenon, the Great Pyramid) are almost always coincidence or motivated reasoning.** The dimensions are close to  $\varphi$  if you pick the right measurements; they fail the test if you pick others.
- **The human body is not actually built on Fibonacci proportions.** Da Vinci's Vitruvian Man uses simple integer ratios (1:8 for head to body, for instance), not  $\varphi$ . Modern measurements of finger bones, the navel-to-foot ratio, and similar claims do not reliably match the golden ratio.

Related study notes: [Whole Numbers](#), [Set](#), [Complex Numbers](#).

The related [OpenStax mathematics textbooks](#) provide additional explanations and exercises for the mathematical definitions used here.

## Recurrence and Indexing

With  $F_0 = 0$ ,  $F_1 = 1$ , the next terms are 1, 2, 3, 5, 8 and 13. The value  $F_6 = 8$  differs from the sixth displayed value if you omit the initial zero when counting a list. Writing the index next to the value avoids that confusion.

The consecutive ratios  $8/5$  and  $13/8$  are 1.6 and 1.625. They lie on opposite sides of the golden ratio and approach it as the indices grow. A tiling made from Fibonacci squares uses integer side lengths; the arcs usually drawn through those squares are circular arcs, so they should not be confused with an exact logarithmic golden spiral.

The points to check in this example are:

- The recurrence applies only after the 2 initial values are fixed.
- Different starting values can follow the same recurrence without being the standard Fibonacci

sequence.

- A picture of square arcs illustrates growth but is not itself a proof of the limiting ratio.

## Check Your Understanding

Try the questions before reading the answers.

**Question 1.** What is  $F_8$  under the stated convention?

**Answer.** 21.

**Question 2.** What follows 21 and 34?

**Answer.** 55.

## Practice Questions

Work each question before reading its solution. The set runs from direct recall and substitution to the applied questions that exams actually use to separate grades. The downloadable PDF includes these questions and their solutions.

**Question 1.** Write the first 10 Fibonacci numbers, starting from  $F_1 = 1, F_2 = 1$ .

**Solution.** 1, 1, 2, 3, 5, 8, 13, 21, 34, 55. Each term is the sum of the 2 before it:  $F_n = F_{n-1} + F_{n-2}$ .

**Question 2.** Compute  $F_{15}$ .

**Solution.** Continue the chain: 55, 89, 144, 233, 377, 610. So  $F_{11} = 89, F_{12} = 144, F_{13} = 233, F_{14} = 377, F_{15} = 610$ .

**Question 3.** Compute the ratio  $F_{10}/F_9$  and compare it with the golden ratio.

**Solution.**  $F_{10}/F_9 = 55/34 \approx 1.6176$ , already close to  $\varphi = (1+\sqrt{5})/2 \approx 1.6180$ . The ratios of consecutive Fibonacci numbers converge to  $\varphi$ , alternating slightly above and below it.

**Question 4.** Verify the identity  $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$  for  $n = 6$ .

**Solution.** Left side:  $1 + 1 + 2 + 3 + 5 + 8 = 20$ . Right side:  $F_8 - 1 = 21 - 1 = 20$ . The identity holds, and it means partial sums of the sequence are themselves almost Fibonacci numbers.

**Question 5.** Use Binet's formula  $F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$  with  $\varphi \approx 1.618, \psi \approx -0.618$  to estimate  $F_6$ .

**Solution.**  $\varphi^6 \approx 17.944$  and  $\psi^6 \approx 0.056$ , so  $F_6 \approx (17.944 - 0.056)/2.236 = 17.888/2.236 = 8.0$ . Exactly right:  $F_6 = 8$ . Because  $|\psi| < 1$ , the  $\psi^n$  term shrinks fast, so  $F_n$  is just  $\varphi^n/\sqrt{5}$  rounded to the nearest integer.

**Question 6.** Every 3rd Fibonacci number is even: 2, 8, 34, 144. Explain why.

**Solution.** Track parities. Starting odd, odd, the pattern of sums is odd + odd = even, then odd + even = odd, then even + odd = odd, and the cycle odd, odd, even repeats forever. An even term appears exactly every 3rd position.

**Question 7.** A staircase has 5 steps and you can climb 1 or 2 steps at a time. How many distinct ways can you reach the top?

**Solution.** Let  $W_n$  be the ways to climb  $n$  steps. The first move is 1 or 2 steps, so  $W_n = W_{n-1} + W_{n-2}$  with  $W_1 = 1$ ,  $W_2 = 2$ . That gives 1, 2, 3, 5, 8: there are 8 ways. The Fibonacci recurrence appears the moment a count splits by its first move.

**Question 8.** Verify Cassini's identity  $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$  for  $n = 5$ .

**Solution.**  $F_6F_4 - F_5^2 = 8 \times 3 - 5^2 = 24 - 25 = -1$ , and  $(-1)^5 = -1$ . It checks. This near-miss between  $F_{n+1}F_{n-1}$  and  $F_n^2$  is also the engine behind a classic dissection puzzle that seems to create area from nothing.

**Question 9.** Why do sunflower heads and pinecones so often show consecutive Fibonacci numbers of spirals?

**Solution.** New florets pack most efficiently when each appears at the golden angle, about  $137.5^\circ$ , from the previous one, because  $\varphi$  is in a precise sense the number hardest to approximate by fractions. Spiral counts made by such packing come out as consecutive Fibonacci numbers. It is efficient geometry, not numerology.

**Question 10.** Using  $F_n \approx \varphi^n / \sqrt{5}$ , estimate  $F_{20}$ .

**Solution.**  $\varphi^{20} \approx 15,127$ , so  $F_{20} \approx 15,127 / 2.236 \approx 6,765$ . The exact value is 6,765. The approximation is already exact after rounding, which shows how completely  $\varphi$  controls the sequence's growth.

## Frequently Asked Questions

### What is the Fibonacci sequence in simple words?

The Fibonacci sequence is a list of numbers where each number is the sum of the two before it. Starting with 1 and 1, the next term is  $1+1 = 2$ , then  $1+2 = 3$ , then  $2+3 = 5$ , then  $3+5 = 8$ , and so on: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

### Why is the Fibonacci sequence important?

Three reasons. First, it is the canonical example of a recurrence relation, used to teach mathematical induction, generating functions, and dynamic programming. Second, its ratios converge to the golden ratio, which connects it to phyllotaxis, geometric design, and architecture. Third, it shows up in many places in nature and computer science, often unexpectedly.

## **What is the golden ratio connection?**

The ratio of consecutive Fibonacci numbers approaches the golden ratio  $\varphi = (1+\sqrt{5})/2 \approx 1.6180$  as  $n$  grows.  $F(n+1)/F(n) \rightarrow \varphi$ . This is why the Fibonacci spiral closely approximates the golden spiral and why sunflower seed spiral counts are usually consecutive Fibonacci numbers.

## **What is Binet's formula?**

Binet's formula gives a closed-form expression for the  $n$ th Fibonacci number:  $F(n) = (\varphi^n - \psi^n) / \sqrt{5}$ , where  $\varphi = (1+\sqrt{5})/2$  is the golden ratio and  $\psi = (1-\sqrt{5})/2$ . For practical computation,  $F(n) \approx \varphi^n / \sqrt{5}$  rounded to the nearest integer, because  $\psi^n$  shrinks quickly.

## **Who invented the Fibonacci sequence?**

The sequence was known to Indian mathematicians (including Pingala around 200 BCE and Hemachandra around 1150) before Leonardo of Pisa (Fibonacci) introduced it to European mathematics in his 1202 book *Liber Abaci* through a problem about rabbit breeding. The naming reflects Western adoption, not original invention.

## **Does the Fibonacci sequence really appear in nature?**

Yes, but not for mystical reasons. The arrangement of leaves, seeds, and petals (phyllotaxis) often follows Fibonacci numbers because turning by the golden angle between successive structures maximizes light capture and packing efficiency. Sunflowers, pinecones, and pineapples are the textbook examples. But many natural Fibonacci claims (architectural proportions, body ratios) are coincidence or motivated reasoning, not real patterns.