Real Sequences

Definitions, Theorems, and Examples

$$\lim_{n\to\infty} a_n = L$$

Gaurav Tiwari

Real Sequences:
Definitions, Theorems, and Examples

A Comprehensive Study

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"The notion of limit is the foundation upon which the whole of analysis rests."
— Karl Weierstrass

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Abstract

Real sequences form the backbone of real analysis and much of advanced calculus. This monograph provides a comprehensive treatment of the theory of real sequences, beginning with basic definitions and progressing through convergence theory, the Monotone Convergence Theorem, Cauchy sequences, the Bolzano–Weierstrass Theorem, and special sequences and limits. Every major theorem is stated precisely and proved in full detail. Numerous worked examples with detailed solutions illustrate the key concepts. We also cover advanced topics including the Stolz–Cesàro theorem, lim sup and lim inf, recursive sequences, the connection to infinite series, and applications to fixed point theory and metric spaces. Historical notes on the contributions of Cauchy, Bolzano, Weierstrass, and others provide context for the development of these fundamental ideas.

Keywords: Real sequences, convergence, limits, Cauchy sequences, Bolzano–Weierstrass theorem, monotone convergence, subsequences, series, real analysis

1 Introduction

1.1 What Are Sequences?

At its most intuitive level, a *sequence* is an ordered list of numbers. When we write

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$$

we have specified an infinite list where every natural number n is assigned the value $1/n$. Sequences are among the simplest infinite mathematical objects, yet they encode an extraordinary amount of structure. The behaviour of sequences—whether they settle down to a limit, oscillate, or grow without bound—is the starting point for all of analysis.

1.2 Why Sequences Matter

Sequences are fundamental for several reasons:

1. Foundation of calculus. The concepts of limit, continuity, differentiability, and integrability are all defined using sequences or their close relatives (ε-δ definitions are equivalent to sequential characterisations).

2. Approximation. Decimal expansions, Newton's method, power series, and numerical algorithms all produce sequences of approximations converging to a desired quantity.

3. Completeness of $\mathbb{R}$. The real number system is characterised by the property that every Cauchy sequence converges. This completeness axiom distinguishes $\mathbb{R}$ from $\mathbb{Q}$.

4. Bridge to series. An infinite series $\sum a_n$ is nothing but the sequence of partial sums $S_n = a_1 + a_2 + \cdots + a_n$. Understanding sequences is prerequisite to understanding series.

5. Applications. Sequences model discrete dynamical systems, population growth, financial compound interest, iterative algorithms, and much more.

1.3 Historical Overview

The rigorous study of sequences began in the early 19th century, though mathematicians had worked with infinite processes for centuries before.

- Augustin-Louis Cauchy (1789–1857) was the first to define convergence rigorously in his *Cours d'Analyse* (1821). He introduced what we now call Cauchy sequences and established the Cauchy criterion for convergence. Cauchy's insistence on rigour transformed calculus from a collection of clever techniques into a deductive science.

- Bernhard Bolzano (1781–1848), a Bohemian priest and mathematician, independently developed many of the same ideas. The Bolzano–Weierstrass theorem, which guarantees that every bounded sequence has a convergent subsequence, bears his name alongside that of Weierstrass. Bolzano's work was largely unrecognised during his lifetime.

- Karl Weierstrass (1815–1897) brought the ε-δ formalism to its mature form. His lectures at the University of Berlin established the modern foundations of analysis. Weierstrass is often called the "father of modern analysis."

- Richard Dedekind (1831–1916) constructed the real numbers via Dedekind cuts, providing the logical foundation for the completeness property that makes the theory of sequences work.

- Georg Cantor (1845–1918) used sequences of nested intervals and Cauchy sequences to construct the real numbers, connecting sequence theory to set theory and the foundations of mathematics.

Remark 1.1. Before the 19th century, mathematicians like Euler and the Bernoullis freely manipulated infinite series and sequences without worrying about convergence. This occasionally led to paradoxes and contradictions, which motivated the rigorous approach developed by Cauchy and Weierstrass.

2 Definitions and Basic Properties

2.1 The Definition of a Sequence

Definition 2.1 (Real Sequence). A real sequence is a function $f : \mathbb{N} \to \mathbb{R}$, where $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers. We write $f(n) = a_n$ and denote the sequence by $\langle a_n \rangle_{n=1}^{\infty}$, or simply $\langle a_n \rangle$ or $\{a_n\}$.

The value a_n is called the n-th term of the sequence. Although a sequence is formally a function, we usually think of it as the ordered list $a_1, a_2, a_3, \dots$

Remark 2.1. Sometimes sequences are indexed starting from $n = 0$ (e.g., $\langle a_n \rangle_{n=0}^{\infty}$). This is a matter of convenience and does not affect convergence properties.

Example 2.1 (The Harmonic Sequence). Define $a_n = \frac{1}{n}$ for $n \geq 1$. This gives

$$\left\langle \frac{1}{n} \right\rangle = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\rangle$$

This is the harmonic sequence, whose terms approach zero as n increases.

Example 2.2 (Alternating Sequence). The sequence $\langle (-1)^n \rangle = \langle -1, 1, -1, 1, \dots \rangle$ oscillates between -1 and 1. It never settles down to a single value.

Example 2.3 (Geometric Sequence). For a fixed $r \in \mathbb{R}$, the sequence $\langle r^n \rangle$ is a geometric sequence. For $r = 1/2$:

$$\left\langle \frac{1}{2^n} \right\rangle = \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots \right\rangle$$

Example 2.4 (Recursive Definition: Fibonacci Sequence). Define $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. This gives the Fibonacci sequence:

$$\langle 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots \rangle$$

Example 2.5 (Constant Sequence). For any fixed $c \in \mathbb{R}$, the sequence $\langle c, c, c, \dots \rangle$ is called the constant sequence equal to c. It trivially converges to c.

2.2 Range Set and Equality

Definition 2.2 (Range Set). The range set of a sequence $\langle a_n \rangle$ is the set $\{a_n : n \in \mathbb{N}\}$ of all distinct values taken by the sequence.

Example 2.6. The range set of $\langle (-1)^n \rangle$ is the finite set $\{-1, 1\}$, even though the sequence has infinitely many terms. The range set of $\langle 1/n \rangle$ is the infinite set $\{1, 1/2, 1/3, \dots\}$.

Definition 2.3 (Equality of Sequences). Two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ are equal if $a_n = b_n$ for every $n \in \mathbb{N}$.

Remark 2.2. Order matters. The sequences $\langle -1, 1, -1, 1, \dots \rangle$ and $\langle 1, -1, 1, -1, \dots \rangle$ have the same range set $\{-1, 1\}$ but are not equal, since they differ at every index.

2.3 Algebra of Sequences

Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be sequences and $c \in \mathbb{R}$. We define:

- Sum: $\langle a_n + b_n \rangle$ with n-th term $a_n + b_n$.
- Difference: $\langle a_n - b_n \rangle$.
- Product: $\langle a_n \cdot b_n \rangle$.
- Scalar multiple: $\langle c \cdot a_n \rangle$.
- Quotient: $\langle a_n / b_n \rangle$, provided $b_n \neq 0$ for all n.
- Reciprocal: $\langle 1/a_n \rangle$, provided $a_n \neq 0$ for all n.

2.4 Bounded and Unbounded Sequences

Definition 2.4 (Bounded Sequence). A sequence $\langle a_n \rangle$ is:

1. bounded above if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$;
2. bounded below if there exists $m \in \mathbb{R}$ such that $a_n \geq m$ for all $n \in \mathbb{N}$;
3. bounded if it is both bounded above and bounded below; equivalently, there exists $K > 0$ such that $|a_n| \leq K$ for all $n \in \mathbb{N}$.

A sequence that is not bounded is called unbounded.

Definition 2.5 (Supremum and Infimum of a Sequence). If $\langle a_n \rangle$ is bounded above, its supremum $\sup_n a_n$ is the least upper bound of the set $\{a_n : n \in \mathbb{N}\}$. If bounded below, its infimum $\inf_n a_n$ is the greatest lower bound.

Example 2.7. The sequence $\langle 1/n \rangle$ is bounded: $0 < 1/n \leq 1$ for all n. Here $\sup = 1$ and $\inf = 0$ (though 0 is not attained).

Example 2.8. The sequence $\langle n^3 \rangle = \langle 1, 8, 27, 64, \ldots \rangle$ is bounded below by 1 but not bounded above. It is unbounded.

Example 2.9. The sequence $\langle (-1)^n n \rangle = \langle -1, 2, -3, 4, -5, \ldots \rangle$ is unbounded. It oscillates with growing magnitude.

Proposition 2.1. If the range set of a sequence is finite, then the sequence is bounded.

Proof. If $\{a_n : n \in \mathbb{N}\}$ is a finite set, then it has a maximum M and a minimum m. Hence $m \leq a_n \leq M$ for all n. □

2.5 Monotone Sequences

Definition 2.6 (Monotone Sequence). A sequence $\langle a_n \rangle$ is:

1. (monotonically) increasing if $a_{n+1} \geq a_n$ for all n;
2. strictly increasing if $a_{n+1} > a_n$ for all n;
3. (monotonically) decreasing if $a_{n+1} \leq a_n$ for all n;
4. strictly decreasing if $a_{n+1} < a_n$ for all n;
5. monotone if it is either increasing or decreasing.

Example 2.10. $\langle 1/n \rangle$ is strictly decreasing. $\langle n^2 \rangle$ is strictly increasing. $\langle 1 - 1/n \rangle = \langle 0, 1/2, 2/3, 3/4, \ldots \rangle$ is strictly increasing and bounded above by 1.

Remark 2.3. A constant sequence is both increasing and decreasing (but not strictly either).

2.6 Subsequences

Definition 2.7 (Subsequence). Let $\langle a_n \rangle$ be a sequence. If $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of natural numbers, then $\langle a_{n_k} \rangle_{k=1}^{\infty}$ is called a subsequence of $\langle a_n \rangle$.

Example 2.11. $\langle 1, 3, 5, 7, \ldots \rangle$ is a subsequence of $\langle 1, 2, 3, 4, 5, \ldots \rangle$ obtained by taking $n_k = 2k - 1$ (the odd-indexed terms).

Example 2.12. $\langle 1, 1, 1, 1, \ldots \rangle$ is a subsequence of $\langle (-1)^n \rangle = \langle -1, 1, -1, 1, \ldots \rangle$ obtained by taking $n_k = 2k$ (the even-indexed terms).

Remark 2.4. Every sequence is a subsequence of itself (take $n_k = k$). A subsequence of a subsequence is again a subsequence of the original sequence.

3 Convergence of Sequences

3.1 The ε-N Definition of a Limit

Definition 3.1 (Convergence of a Sequence). A sequence $\langle a_n \rangle$ converges to a real number L if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \quad \text{for all } n \geq N.$$

We write $\lim_{n\to\infty} a_n = L$, or $a_n \to L$ as $n \to \infty$, and call L the limit of the sequence. A sequence that converges is called convergent; otherwise it is called non-convergent.

Remark 3.1. The condition $|a_n - L| < \varepsilon$ means $L - \varepsilon < a_n < L + \varepsilon$. Geometrically, all terms from the N-th onward lie in the open interval $(L - \varepsilon, L + \varepsilon)$.

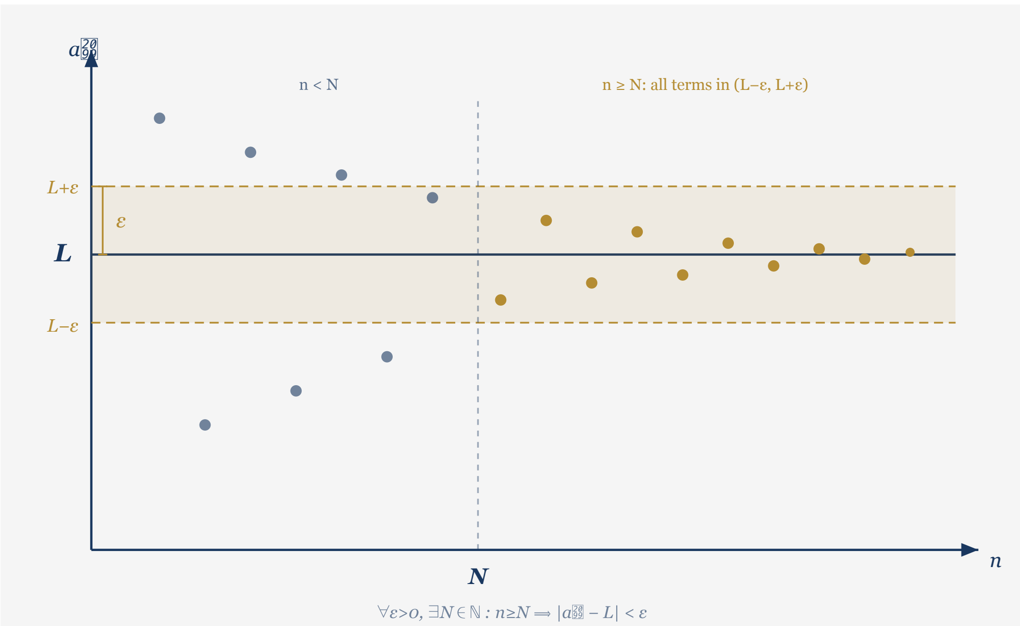

Figure 1: The ε-N definition of convergence: for any $\varepsilon > 0$, all terms beyond index N lie within the band $(L - \varepsilon, L + \varepsilon)$.

Example 3.1 (Proving $\lim_{n\to\infty} \frac{1}{n} = 0$ from the definition). Let $\varepsilon > 0$ be given. We need N such that $|1/n - 0| = 1/n < \varepsilon$ for all $n \geq N$. Choose N to be any natural number with $N > 1/\varepsilon$ (which exists by the Archimedean property). Then for $n \geq N$:

$$\frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence $\lim_{n\to\infty} \frac{1}{n} = 0$. □

Example 3.2 (Proving $\lim_{n\to\infty} \frac{3n+1}{n+2} = 3$). We compute:

$$\left|\frac{3n+1}{n+2} - 3\right| = \left|\frac{3n+1-3(n+2)}{n+2}\right| = \left|\frac{-5}{n+2}\right| = \frac{5}{n+2}.$$

Given $\varepsilon > 0$, we need $\frac{5}{n+2} < \varepsilon$, i.e., $n > \frac{5}{\varepsilon} - 2$. Choose $N > \frac{5}{\varepsilon} - 2$. Then for all $n \geq N$:

$$\left|\frac{3n+1}{n+2} - 3\right| = \frac{5}{n+2} \leq \frac{5}{N+2} < \varepsilon.$$

Example 3.3 (Proving $\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}$). We compute:

$$\left|\frac{n^2}{2n^2+1} - \frac{1}{2}\right| = \left|\frac{2n^2-2n^2-1}{2(2n^2+1)}\right| = \frac{1}{2(2n^2+1)} < \frac{1}{4n^2}.$$

Given $\varepsilon > 0$, choose $N > \frac{1}{2\sqrt{\varepsilon}}$. Then for $n \geq N$: $\frac{1}{4n^2} \leq \frac{1}{4N^2} < \varepsilon$.

3.2 Uniqueness of Limits

Theorem 3.1 (Uniqueness of Limits). If a sequence $\langle a_n \rangle$ converges, its limit is unique.

Proof. Suppose $a_n \to L$ and $a_n \to L'$ with $L \neq L'$. Let $\varepsilon = |L - L'|/2 > 0$. Since $a_n \to L$, there exists N_1 such that $|a_n - L| < \varepsilon$ for all $n \geq N_1$. Since $a_n \to L'$, there exists N_2 such that $|a_n - L'| < \varepsilon$ for all $n \geq N_2$.

For $n \geq \max(N_1, N_2)$, by the triangle inequality:

$$|L - L'| \leq |L - a_n| + |a_n - L'| < \varepsilon + \varepsilon = 2\varepsilon = |L - L'|.$$

This gives $|L - L'| < |L - L'|$, a contradiction. Hence $L = L'$. □

3.3 Every Convergent Sequence Is Bounded

Theorem 3.2. Every convergent sequence is bounded.

Proof. Let $a_n \to L$. Taking $\varepsilon = 1$, there exists N such that $|a_n - L| < 1$ for all $n \geq N$. Then $|a_n| < |L| + 1$ for $n \geq N$. Define

$$K = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, |L| + 1\}.$$

Then $|a_n| \leq K$ for all $n \in \mathbb{N}$. □

Remark 3.2. The converse is false: $\langle (-1)^n \rangle$ is bounded but not convergent.

3.4 Algebra of Limits

Theorem 3.3 (Algebra of Limits). Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be convergent sequences with $\lim a_n = L$ and $\lim b_n = M$. Then:

1. $\lim(a_n + b_n) = L + M$;
2. $\lim(a_n - b_n) = L - M$;
3. $\lim(c \cdot a_n) = c \cdot L$ for any constant $c \in \mathbb{R}$;

4. $\lim(a_n \cdot b_n) = L \cdot M$;
5. $\lim \frac{a_n}{b_n} = \frac{L}{M}$, provided $M \neq 0$ and $b_n \neq 0$ for all n.

Proof of (1). Let $\varepsilon > 0$. There exist N_1, N_2 such that $|a_n - L| < \varepsilon/2$ for $n \geq N_1$ and $|b_n - M| < \varepsilon/2$ for $n \geq N_2$. For $n \geq N = \max(N_1, N_2)$:

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Proof of (4). We write $a_n b_n - LM = a_n(b_n - M) + M(a_n - L)$. Since $\langle a_n \rangle$ converges, it is bounded: $|a_n| \leq K$ for some $K > 0$ and all n. Given $\varepsilon > 0$, choose N_1 such that $|b_n - M| < \frac{\varepsilon}{2K}$ for $n \geq N_1$, and N_2 such that $|a_n - L| < \frac{\varepsilon}{2(|M|+1)}$ for $n \geq N_2$. Then for $n \geq \max(N_1, N_2)$:

$$|a_n b_n - LM| \leq |a_n||b_n - M| + |M||a_n - L| < K \cdot \frac{\varepsilon}{2K} + |M| \cdot \frac{\varepsilon}{2(|M| + 1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Proof of (5). It suffices to show $\lim \frac{1}{b_n} = \frac{1}{M}$, then apply (4). Since $M \neq 0$ and $b_n \to M$, there exists N_0 such that $|b_n| > |M|/2$ for $n \geq N_0$. Then:

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|M - b_n|}{|b_n||M|} < \frac{|b_n - M|}{|M|^2/2} = \frac{2|b_n - M|}{|M|^2}.$$

Given $\varepsilon > 0$, choose $N_1 \geq N_0$ such that $|b_n - M| < \frac{\varepsilon|M|^2}{2}$ for $n \geq N_1$. Then $|1/b_n - 1/M| < \varepsilon$. □

3.5 The Squeeze Theorem

Theorem 3.4 (Squeeze Theorem / Sandwich Theorem). Let $\langle a_n \rangle, \langle b_n \rangle, \langle c_n \rangle$ be sequences such that

$$a_n \leq b_n \leq c_n \quad \text{for all } n \geq N_0$$

for some $N_0 \in \mathbb{N}$. If $\lim a_n = L = \lim c_n$, then $\lim b_n = L$.

Proof. Let $\varepsilon > 0$. There exist N_1, N_2 such that $|a_n - L| < \varepsilon$ for $n \geq N_1$ and $|c_n - L| < \varepsilon$ for $n \geq N_2$. Let $N = \max(N_0, N_1, N_2)$. For $n \geq N$:

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon,$$

so $|b_n - L| < \varepsilon$. □

> Squeeze Theorem: If $a_n \leq b_n \leq c_n$ for large n, and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.
> This is one of the most frequently used tools for computing limits.

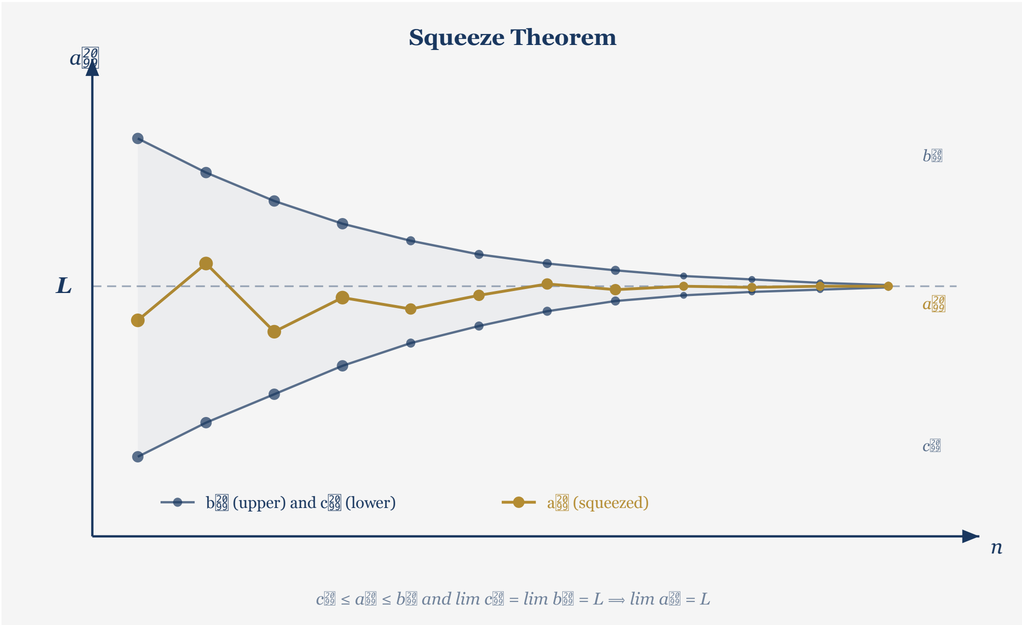

Figure 2: The Squeeze Theorem: the sequence $\langle b_n \rangle$ is trapped between $\langle a_n \rangle$ and $\langle c_n \rangle$, both converging to L, forcing $b_n \to L$.

Example 3.4 (Application of the Squeeze Theorem). Show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

Solution. Since $-1 \le \sin n \le 1$ for all n, we have

$$\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}.$$

Since $\lim(-1/n) = 0 = \lim(1/n)$, the Squeeze Theorem gives $\lim \frac{\sin n}{n} = 0$.

Example 3.5. Show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Solution. For $n \ge 1$:

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n}{n} \le \frac{1}{n} \cdot 1 \cdot 1 \cdots 1 = \frac{1}{n}.$$

Since $0 \le \frac{n!}{n^n} \le \frac{1}{n}$ and $\lim \frac{1}{n} = 0$, the result follows.

Example 3.6. Show that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Solution. We have $\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n}$. Given $\varepsilon > 0$, choose $N > 1/\varepsilon$. Then for $n \ge N$: $\frac{1}{n} \le \frac{1}{N} < \varepsilon$. Hence $\frac{(-1)^n}{n} \to 0$.

Example 3.7. Prove that the sequence $a_n = \frac{2n^2+3}{n^2+5n}$ converges and find its limit.

Solution. Divide numerator and denominator by n^2:

$$a_n = \frac{2 + 3/n^2}{1 + 5/n}.$$

Since $3/n^2 \to 0$ and $5/n \to 0$, the algebra of limits gives $a_n \to \frac{2+0}{1+0} = 2$.

Example 3.8. Show that $\lim_{n\to\infty} \frac{\sqrt{n+1}-\sqrt{n}}{1} = 0$.

Solution. Rationalise:

$$\sqrt{n+1} - \sqrt{n} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \le \frac{1}{2\sqrt{n}} \to 0.$$

3.6 Order Preservation and Non-Negative Limits

Theorem 3.5 (Order Preservation). If $a_n \leq b_n$ for all $n \geq N_0$ and both sequences converge, then $\lim a_n \leq \lim b_n$.

Proof. Let $L = \lim a_n$ and $M = \lim b_n$. Define $c_n = b_n - a_n \geq 0$ for $n \geq N_0$. Then $\lim c_n = M - L$. If $M - L < 0$, then there exists N such that $c_n < 0$ for $n \geq N$, contradicting $c_n \geq 0$. Hence $M - L \geq 0$, i.e., $L \leq M$. $\square$

Corollary 3.6. If $a_n \geq 0$ for all n and $\langle a_n \rangle$ converges, then $\lim a_n \geq 0$.

Remark 3.3. Strict inequality is not preserved: $1/n > 0$ for all n, but $\lim 1/n = 0$, not > 0.

3.7 Absolute Value and Convergence

Theorem 3.7. If $\lim a_n = L$, then $\lim |a_n| = |L|$. The converse holds when $L = 0$: if $\lim |a_n| = 0$, then $\lim a_n = 0$.

Proof. By the reverse triangle inequality: $||a_n| - |L|| \leq |a_n - L|$. Given $\varepsilon > 0$, choose N such that $|a_n - L| < \varepsilon$ for $n \geq N$. Then $||a_n| - |L|| < \varepsilon$.

For the partial converse: if $\lim |a_n| = 0$, then $|a_n - 0| = |a_n| = ||a_n| - 0| \to 0$. $\square$

Remark 3.4. The converse fails when $L \neq 0$: $|(-1)^n| = 1 \to 1$, but $(-1)^n$ does not converge.

3.8 Convergence of Subsequences

Theorem 3.8. If $\langle a_n \rangle$ converges to L, then every subsequence $\langle a_{n_k} \rangle$ also converges to L.

Proof. Let $\varepsilon > 0$. There exists N such that $|a_n - L| < \varepsilon$ for all $n \geq N$. Since $n_1 < n_2 < \cdots$ with each $n_k \in \mathbb{N}$, we have $n_k \geq k$ for all k (by induction). Choose $K = N$. For $k \geq K$: $n_k \geq k \geq N$, so $|a_{n_k} - L| < \varepsilon$. $\square$

Subsequence Divergence Test: If two subsequences of $\langle a_n \rangle$ converge to different limits, then $\langle a_n \rangle$ diverges.

Example 3.9. The sequence $\langle (-1)^n \rangle$ has the subsequence $\langle a_{2k} \rangle = \langle 1, 1, 1, \ldots \rangle \to 1$ and $\langle a_{2k-1} \rangle = \langle -1, -1, -1, \ldots \rangle \to -1$. Since $1 \neq -1$, the sequence $\langle (-1)^n \rangle$ does not converge.

4 The Monotone Convergence Theorem

4.1 Statement and Proof

Theorem 4.1 (Monotone Convergence Theorem).

1. A monotonically increasing sequence that is bounded above converges to its supremum.
2. A monotonically decreasing sequence that is bounded below converges to its infimum.

Proof. We prove (1); the proof of (2) is analogous.

Let $\langle a_n \rangle$ be increasing and bounded above. The set $S = \{a_n : n \in \mathbb{N}\}$ is non-empty and bounded above. By the completeness axiom of $\mathbb{R}$, $L = \sup S$ exists.

We claim $a_n \to L$. Let $\varepsilon > 0$. Since $L = \sup S$ and $L - \varepsilon$ is not an upper bound of S, there exists N such that $a_N > L - \varepsilon$. Since $\langle a_n \rangle$ is increasing, for $n \geq N$:

$$L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon.$$

Hence $|a_n - L| < \varepsilon$ for all $n \geq N$. $\square$

Corollary 4.2. A monotone sequence converges if and only if it is bounded.

Proof. If it converges, it is bounded (Theorem 3.2). Conversely, a bounded monotone sequence converges by Theorem 4.1. □

> Monotone Convergence Theorem: A bounded monotone sequence converges.
> This is one of the most powerful tools for proving convergence. You only need to check two things: monotonicity and boundedness.

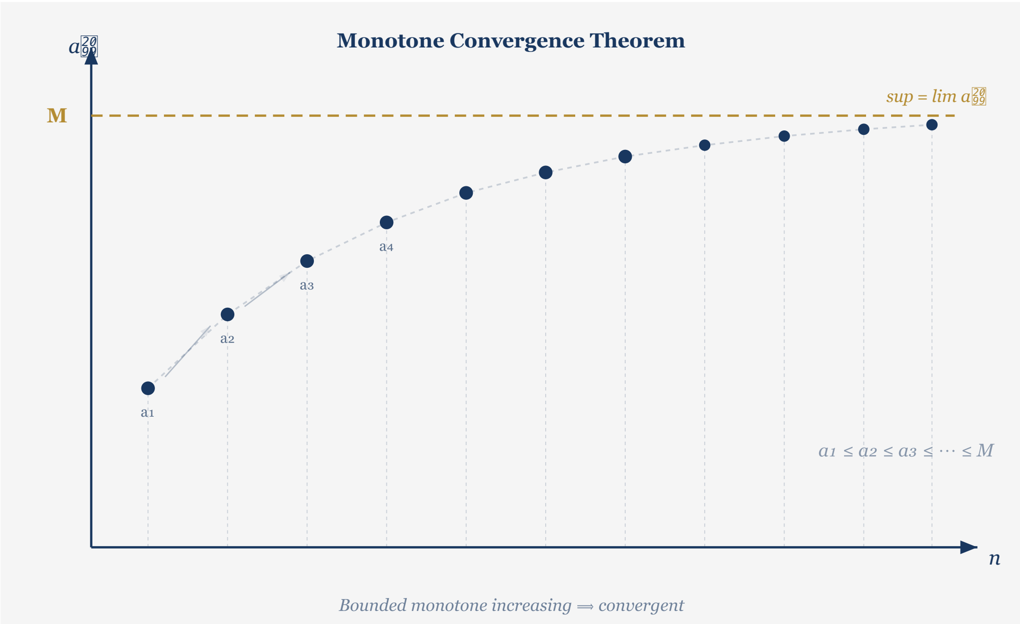

Figure 3: The Monotone Convergence Theorem: a bounded, increasing sequence converges to its supremum $L = \sup\{a_n\}$.

4.2 Applications of the Monotone Convergence Theorem

Example 4.1 (The sequence $a_n = (1 + 1/n)^n$). Define $a_n = \left(1 + \frac{1}{n}\right)^n$. We shall show this sequence is increasing and bounded above, hence convergent. Its limit is the number e.

Monotonicity. By the AM–GM inequality applied to n copies of $(1 + 1/n)$ and one copy of 1:

$$\frac{n \cdot \left(1 + \frac{1}{n}\right) + 1}{n + 1} \geq \left(\left(1 + \frac{1}{n}\right)^n \cdot 1\right)^{1/(n+1)}$$

which simplifies to show $a_{n+1} \geq a_n$ (the full verification uses the binomial theorem and is a standard exercise).

Boundedness. By the binomial theorem:

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \leq 1 + \sum_{k=1}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} < 1 + 2 = 3.$$

So $a_n < 3$ for all n.

By the Monotone Convergence Theorem, $\langle a_n \rangle$ converges. We define $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828$.

Example 4.2 (A recursively defined sequence). Define $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.

Claim: $\langle a_n \rangle$ converges to 2.

Step 1: Boundedness. We prove $a_n < 2$ for all n by induction. Base case: $a_1 = 1 < 2$. Inductive step: if $a_n < 2$, then $a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = 2$.

Step 2: Monotonicity. We show $a_{n+1} > a_n$ for all n by induction. Base case: $a_2 = \sqrt{3} \approx 1.732 > 1 = a_1$. Inductive step: if $a_n > a_{n-1}$, then $a_{n+1} = \sqrt{2 + a_n} > \sqrt{2 + a_{n-1}} = a_n$.

Step 3: Finding the limit. By MCT, $L = \lim a_n$ exists. Taking limits in $a_{n+1} = \sqrt{2 + a_n}$:

$$L = \sqrt{2 + L} \implies L^2 = 2 + L \implies L^2 - L - 2 = 0 \implies (L - 2)(L + 1) = 0.$$

Since $a_n > 0$ for all n, we have $L \geq 0$, so $L = 2$.

Example 4.3 (Nested radical sequence). Define $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 \cdot a_n}$ for $n \geq 1$. Then $a_n = 2^{1-2^{-n}}$, and $\lim a_n = 2$.

To see this directly: the sequence is increasing (since $a_1 = \sqrt{2} < 2 = \sqrt{2 \cdot 2} > \sqrt{2 \cdot \sqrt{2}} = a_2$ requires checking). Alternatively, $a_n = 2^{1-1/2^n} \to 2^1 = 2$.

Example 4.4 (Computing $\sqrt{c}$ by iteration). The Babylonian method (or Heron's method) for computing $\sqrt{c}$ ($c > 0$) defines:

$$a_1 > 0, \qquad a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right).$$

By AM–GM, $a_{n+1} = \frac{1}{2}(a_n + c/a_n) \geq \sqrt{c}$ for all $n \geq 1$, so $a_n \geq \sqrt{c}$ for $n \geq 2$. Also $a_{n+1} \leq a_n$ for $n \geq 2$ (since $a_n \geq \sqrt{c}$). The sequence is eventually decreasing and bounded below by $\sqrt{c}$, so it converges. If L is the limit:

$$L = \frac{1}{2}\left(L + \frac{c}{L}\right) \implies 2L^2 = L^2 + c \implies L^2 = c \implies L = \sqrt{c}.$$

5 Cauchy Sequences

5.1 Definition

Definition 5.1 (Cauchy Sequence). A sequence $\langle a_n \rangle$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \quad \text{for all } n, m \geq N.$$

The key idea: the terms of a Cauchy sequence become arbitrarily close to *each other*, without reference to any particular limit.

5.2 Every Convergent Sequence Is Cauchy

Theorem 5.1. Every convergent sequence is a Cauchy sequence.

Proof. Let $a_n \to L$ and $\varepsilon > 0$. There exists N such that $|a_n - L| < \varepsilon/2$ for all $n \geq N$. For $n, m \geq N$:

$$|a_n - a_m| \leq |a_n - L| + |L - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

5.3 Every Cauchy Sequence Is Bounded

Theorem 5.2. Every Cauchy sequence is bounded.

Proof. Let $\langle a_n \rangle$ be Cauchy. Taking $\varepsilon = 1$, there exists N such that $|a_n - a_m| < 1$ for all $n, m \geq N$. In particular, $|a_n - a_N| < 1$ for $n \geq N$, so $|a_n| < |a_N| + 1$. Set $K = \max\{|a_1|, \ldots, |a_{N-1}|, |a_N| + 1\}$. Then $|a_n| \leq K$ for all n. $\square$

5.4 Completeness of $\mathbb{R}$: The Cauchy Criterion

Theorem 5.3 (Cauchy's General Principle of Convergence). A sequence of real numbers converges if and only if it is a Cauchy sequence.

Proof. ($\Rightarrow$) This is Theorem 5.1.

($\Leftarrow$) Let $\langle a_n \rangle$ be Cauchy. By Theorem 5.2, it is bounded. By the Bolzano–Weierstrass Theorem (Theorem 6.1 below), it has a convergent subsequence $a_{n_k} \to L$.

We show $a_n \to L$. Let $\varepsilon > 0$. Since $\langle a_n \rangle$ is Cauchy, there exists N_1 such that $|a_n - a_m| < \varepsilon/2$ for $n, m \geq N_1$. Since $a_{n_k} \to L$, there exists K such that $|a_{n_k} - L| < \varepsilon/2$ for $k \geq K$.

Choose k large enough that $k \geq K$ and $n_k \geq N_1$. For $n \geq N_1$:

$$|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Cauchy Criterion: A sequence in $\mathbb{R}$ converges $\iff$ it is Cauchy.
This is a manifestation of the completeness of $\mathbb{R}$. The rational numbers $\mathbb{Q}$ are not complete: the sequence $1, 1.4, 1.41, 1.414, \ldots$ (decimal approximations of $\sqrt{2}$) is Cauchy in $\mathbb{Q}$ but does not converge in $\mathbb{Q}$.

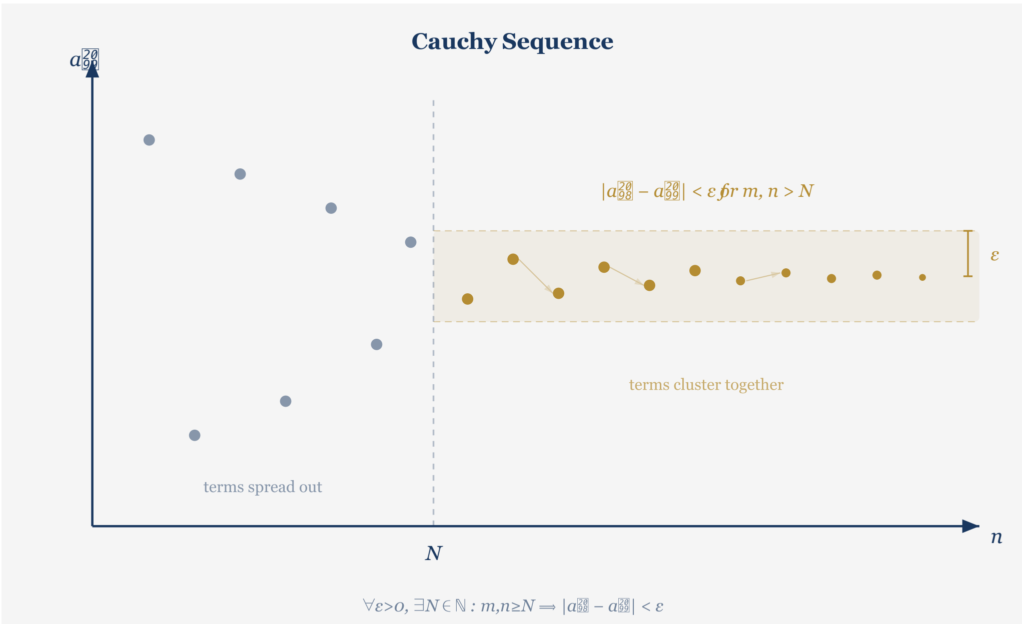

Figure 4: A Cauchy sequence: for any $\varepsilon > 0$, all terms beyond some index N lie within ε of each other, clustering ever more tightly.

Remark 5.1. The Cauchy criterion is invaluable because it allows us to prove convergence without knowing the limit. This is particularly useful for sequences defined implicitly or by complicated recursions.

Example 5.1. Show that $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is a Cauchy sequence.

Solution. For $n > m$:

$$|a_n - a_m| = \sum_{k=m+1}^{n} \frac{1}{k^2} < \sum_{k=m+1}^{n} \frac{1}{k(k-1)} = \sum_{k=m+1}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) = \frac{1}{m} - \frac{1}{n} < \frac{1}{m}.$$

Given $\varepsilon > 0$, choose $N > 1/\varepsilon$. Then for $n > m \geq N$: $|a_n - a_m| < 1/m \leq 1/N < \varepsilon$.

Hence $\langle a_n \rangle$ is Cauchy, and therefore converges. (The limit turns out to be $\pi^2/6$, a famous result due to Euler.)

6 Subsequences and the Bolzano–Weierstrass Theorem

6.1 Limit Points of a Sequence

Definition 6.1 (Limit Point / Accumulation Point). A real number P is a limit point (or accumulation point, or cluster point) of a sequence $\langle a_n \rangle$ if for every $\varepsilon > 0$, the interval $(P - \varepsilon, P + \varepsilon)$ contains a_n for infinitely many values of n.

Equivalently, P is a limit point if and only if there exists a subsequence $\langle a_{n_k} \rangle$ converging to P.

Example 6.1. The sequence $\langle (-1)^n \rangle$ has two limit points: -1 and 1. The subsequence of odd-indexed terms converges to -1, and the subsequence of even-indexed terms converges to 1.

Example 6.2. The sequence $\langle (-1)^n(1 + 1/n) \rangle = \langle -2, 3/2, -4/3, 5/4, \ldots \rangle$ has limit points -1 and 1.

Example 6.3. Consider the sequence where a_n enumerates the rationals in $(0, 1)$ (this is possible since $\mathbb{Q} \cap (0, 1)$ is countable). Every point in $[0, 1]$ is a limit point of this sequence.

6.2 The Bolzano–Weierstrass Theorem

Theorem 6.1 (Bolzano–Weierstrass). Every bounded sequence of real numbers has at least one limit point. Equivalently, every bounded sequence has a convergent subsequence.

Proof. Let $\langle a_n \rangle$ be bounded, say $a_n \in [a, b]$ for all n. We construct a convergent subsequence using the nested intervals method (bisection).

Step 1. Set $[a_1^*, b_1^*] = [a, b]$. Bisect the interval into $[a, (a + b)/2]$ and $[(a + b)/2, b]$. At least one of these half-intervals contains infinitely many terms of the sequence (since the full interval contains all terms, and the union of two finite sets is finite). Choose such a half-interval and call it $[a_2^*, b_2^*]$. Pick n_1 such that $a_{n_1} \in [a_2^*, b_2^*]$.

Step 2. Bisect $[a_2^*, b_2^*]$ into two equal parts. Again, at least one half contains infinitely many terms. Choose it as $[a_3^*, b_3^*]$ and pick $n_2 > n_1$ with $a_{n_2} \in [a_3^*, b_3^*]$.

Continue inductively. At stage k, we have an interval $[a_k^*, b_k^*]$ of length $(b - a)/2^{k-1}$ containing $a_{n_{k-1}}$, and we pick $n_k > n_{k-1}$ with $a_{n_k} \in [a_{k+1}^*, b_{k+1}^*]$.

The nested intervals $[a_k^*, b_k^*]$ satisfy:

- $[a_{k+1}^*, b_{k+1}^*] \subseteq [a_k^*, b_k^*]$ for all k;
- $b_k^* - a_k^* = (b - a)/2^{k-1} \to 0$.

By the Nested Intervals Property of $\mathbb{R}$, there exists a unique $L \in \bigcap_{k=1}^{\infty} [a_k^*, b_k^*]$. Since $a_{n_k} \in [a_k^*, b_k^*]$ and $b_k^* - a_k^* \to 0$, we have $a_{n_k} \to L$. $\square$

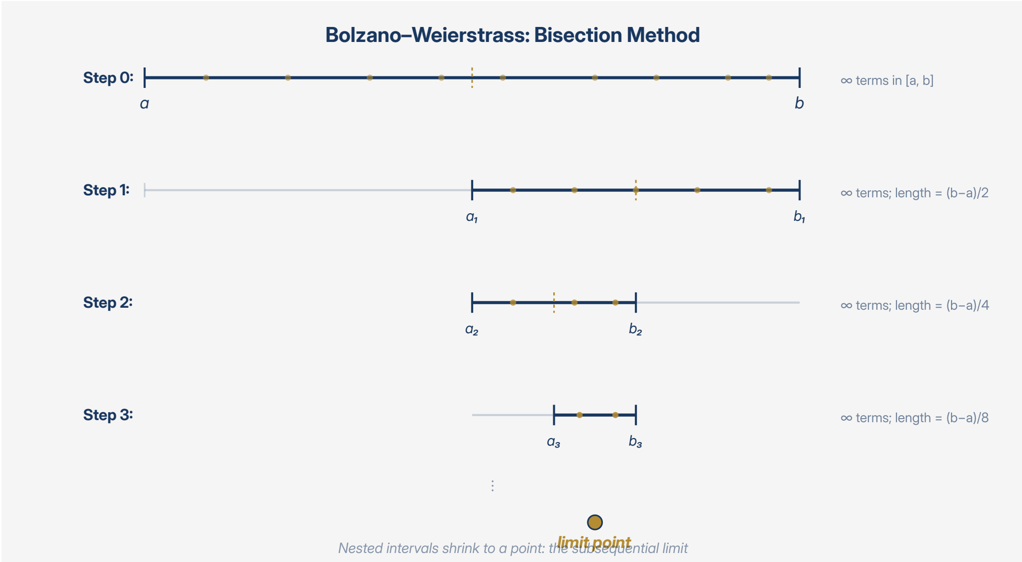

Figure 5: The Bolzano–Weierstrass Theorem via bisection: repeated halving of the interval produces nested subintervals, each containing infinitely many terms, converging to a limit point.

Remark 6.1. An unbounded sequence need not have any limit point (e.g., $\langle n \rangle$), but it may (e.g., $\langle 0, 1, 0, 2, 0, 3, 0, 4, \dots \rangle$ is unbounded but has 0 as a limit point).

6.3 Limit Superior and Limit Inferior

Definition 6.2 (lim sup and lim inf). Let $\langle a_n \rangle$ be a bounded sequence. Define:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup_{k\geq n} a_k \right) = \inf_{n\geq 1} \sup_{k\geq n} a_k, \tag{1}$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \left(\inf_{k\geq n} a_k \right) = \sup_{n\geq 1} \inf_{k\geq n} a_k. \tag{2}$$

Theorem 6.2. Let $\langle a_n \rangle$ be a bounded sequence. Then:

1. $\limsup a_n$ is the largest limit point of $\langle a_n \rangle$.
2. $\liminf a_n$ is the smallest limit point of $\langle a_n \rangle$.
3. $\liminf a_n \leq \limsup a_n$.
4. $\langle a_n \rangle$ converges if and only if $\limsup a_n = \liminf a_n$, and in that case the common value is $\lim a_n$.

Proof of (4). ($\Rightarrow$) If $a_n \to L$, then for any $\varepsilon > 0$, eventually $L - \varepsilon < a_n < L + \varepsilon$. So $\sup_{k\geq n} a_k \leq L + \varepsilon$ and $\inf_{k\geq n} a_k \geq L - \varepsilon$ for large n. Hence $\limsup a_n \leq L + \varepsilon$ and $\liminf a_n \geq L - \varepsilon$. Since ε is arbitrary, $\limsup a_n = \liminf a_n = L$.

($\Leftarrow$) If $\limsup a_n = \liminf a_n = L$, then $\sup_{k\geq n} a_k \to L$ and $\inf_{k\geq n} a_k \to L$. Since $\inf_{k\geq n} a_k \leq a_n \leq \sup_{k\geq n} a_k$, the Squeeze Theorem gives $a_n \to L$. □

Example 6.4. For $\langle (-1)^n \rangle$: $\limsup = 1$ and $\liminf = -1$. Since these are unequal, the sequence diverges.

Example 6.5. For $a_n = (-1)^n/n$: $\sup_{k\geq n} a_k \to 0$ and $\inf_{k\geq n} a_k \to 0$. So $\limsup = \liminf = 0$, confirming $a_n \to 0$.

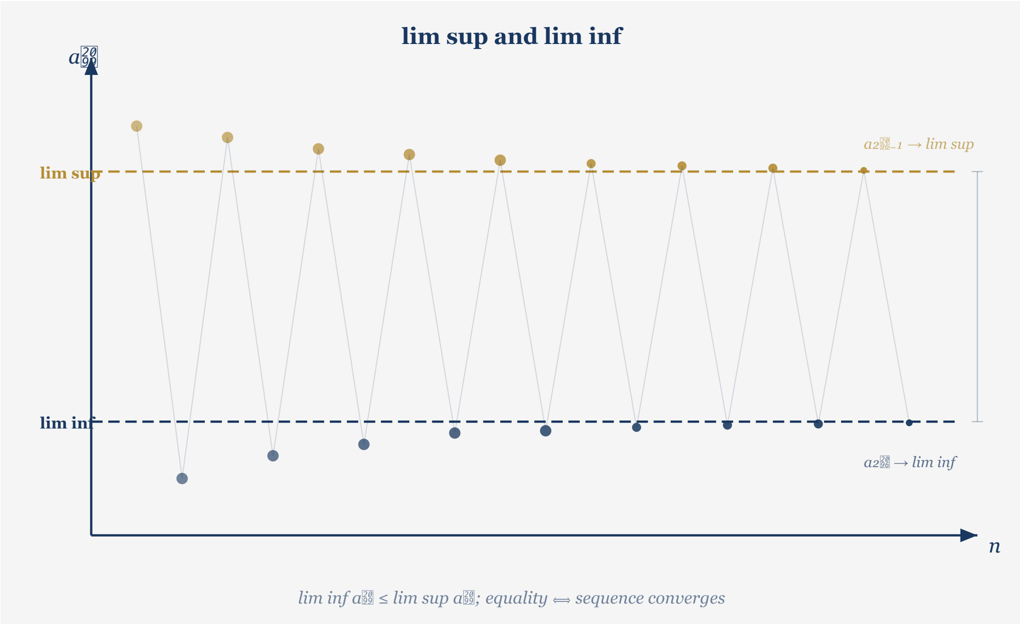

Figure 6: Visualising lim sup and lim inf: the sequences $\sup_{k\geq n} a_k$ and $\inf_{k\geq n} a_k$ form decreasing and increasing envelopes, respectively, that squeeze the original sequence.

7 Special Sequences and Limits

7.1 Important Standard Limits

The following limits appear repeatedly throughout analysis. Each is worth knowing by heart.

Standard Limits:

1. $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for any $p > 0$.
2. $\lim_{n\to\infty} \sqrt[n]{a} = 1$ for any $a > 0$.
3. $\lim_{n\to\infty} n^{1/n} = 1$.
4. $\lim_{n\to\infty} \frac{n^k}{a^n} = 0$ for any fixed k and $a > 1$ (exponentials dominate polynomials).
5. $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for any $a \in \mathbb{R}$ (factorials dominate exponentials).
6. $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.
7. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x \in \mathbb{R}$.
8. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

7.2 Proof That $\lim n^{1/n} = 1$

Theorem 7.1. $\lim_{n\to\infty} n^{1/n} = 1$.

Proof. For $n \geq 2$, write $n^{1/n} = 1 + h_n$ where $h_n > 0$. Then $n = (1+h_n)^n$. By the binomial theorem:

$$n = (1+h_n)^n \geq 1 + \binom{n}{2}h_n^2 = 1 + \frac{n(n-1)}{2}h_n^2.$$

Hence $n - 1 \geq \frac{n(n-1)}{2}h_n^2$, so $h_n^2 \leq \frac{2}{n}$, giving $0 < h_n \leq \sqrt{2/n}$.

Since $\sqrt{2/n} \to 0$, the Squeeze Theorem gives $h_n \to 0$, hence $n^{1/n} = 1 + h_n \to 1$. □

7.3 Proof That $a^{1/n} \to 1$ for $a > 0$

Theorem 7.2. For any $a > 0$, $\lim_{n\to\infty} a^{1/n} = 1$.

Proof. Case 1: $a \geq 1$. Write $a^{1/n} = 1 + h_n$ with $h_n \geq 0$. Then $a = (1+h_n)^n \geq 1 + nh_n$, so $0 \leq h_n \leq (a-1)/n \to 0$.

Case 2: $0 < a < 1$. Then $1/a > 1$, and $(1/a)^{1/n} \to 1$ by Case 1. Since $a^{1/n} = 1/(1/a)^{1/n}$, we get $a^{1/n} \to 1/1 = 1$. □

7.4 Exponentials Dominate Polynomials

Theorem 7.3. For any fixed $k \in \mathbb{N}$ and $a > 1$: $\lim_{n\to\infty} \frac{n^k}{a^n} = 0$.

Proof. Write $a = 1 + h$ with $h > 0$. For $n > 2k$:

$$a^n = (1+h)^n \geq \binom{n}{2k}h^{2k} \geq \frac{n^{2k}}{(2k)! \cdot 2^{2k}}h^{2k}$$

(using $\binom{n}{2k} \geq (n/2k)^{2k}/(2k)!$ is rough; more carefully, $\binom{n}{2k} \geq \frac{(n-2k)^{2k}}{(2k)!}$). Therefore:

$$0 \leq \frac{n^k}{a^n} \leq \frac{(2k)!}{h^{2k}} \cdot \frac{n^k}{(n-2k)^{2k}} \to 0$$

since the exponent $2k$ in the denominator exceeds k. □

7.5 Factorials Dominate Exponentials

Theorem 7.4. For any $a \in \mathbb{R}$: $\lim_{n\to\infty} \frac{a^n}{n!} = 0$.

Proof. We may assume $a > 0$ (otherwise consider $|a|$). Choose $N \in \mathbb{N}$ with $N > 2a$. For $n > N$:

$$\left|\frac{a^n}{n!}\right| = \frac{a^N}{N!} \cdot \frac{a}{N+1} \cdot \frac{a}{N+2} \cdots \frac{a}{n} \leq \frac{a^N}{N!} \cdot \left(\frac{1}{2}\right)^{n-N} \to 0$$

since each factor $a/k \leq 1/2$ for $k > N > 2a$. □

7.6 Euler's Number e

Theorem 7.5. The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is increasing and bounded above by 3. Its limit is denoted e.

The proof was given in Example 4.1. The number $e \approx 2.71828\ldots$ is one of the most important constants in mathematics.

Theorem 7.6. More generally, $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for every $x \in \mathbb{R}$.

7.7 Cauchy's Theorems on Limits

Theorem 7.7 (Cauchy's First Theorem on Limits / Cesàro Mean). If $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = L.$$

Proof. Let $\varepsilon > 0$. Choose N such that $|a_n - L| < \varepsilon/2$ for $n \geq N$. Then:

$$\frac{a_1 + \cdots + a_n}{n} - L = \frac{(a_1 - L) + \cdots + (a_N - L) + (a_{N+1} - L) + \cdots + (a_n - L)}{n}.$$

The first N terms contribute a fixed sum $S_N = \sum_{k=1}^{N}(a_k - L)$. For $n > N$:

$$\left|\frac{a_1 + \cdots + a_n}{n} - L\right| \leq \frac{|S_N|}{n} + \frac{(n-N)\cdot\varepsilon/2}{n} < \frac{|S_N|}{n} + \frac{\varepsilon}{2}.$$

Choose n large enough that $|S_N|/n < \varepsilon/2$. Then the total is less than ε. □

Remark 7.1. The converse of the Cesàro mean theorem is false. The sequence $\langle(-1)^n\rangle$ has Cesàro mean tending to 0, but the sequence itself does not converge.

Theorem 7.8 (Cauchy's Second Theorem on Limits). If $\langle a_n\rangle$ is a sequence of positive real numbers with $\lim a_n = L > 0$, then

$$\lim_{n\to\infty} (a_1 \cdot a_2 \cdots a_n)^{1/n} = L.$$

Proof. Apply Theorem 7.7 to the sequence $\langle \ln a_n\rangle$, which converges to $\ln L$. Then

$$\frac{\ln a_1 + \cdots + \ln a_n}{n} \to \ln L,$$

so $(a_1 \cdots a_n)^{1/n} = \exp\left(\frac{\ln a_1 + \cdots + \ln a_n}{n}\right) \to e^{\ln L} = L.$ □

7.8 The Stolz–Cesàro Theorem

Theorem 7.9 (Stolz–Cesàro). Let $\langle b_n\rangle$ be a strictly increasing sequence with $b_n \to +\infty$. If

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = L \quad (\text{where } L \in \mathbb{R} \cup \{\pm\infty\}),$$

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

Remark 7.2. The Stolz–Cesàro theorem is the discrete analogue of L'Hôpital's rule. It is extremely useful for evaluating limits of the form a_n/b_n where $b_n \to \infty$.

Example 7.1. Compute $\lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3}$.

Solution. Set $a_n = 1^2 + 2^2 + \cdots + n^2$ and $b_n = n^3$. Then $a_n - a_{n-1} = n^2$ and $b_n - b_{n-1} = n^3 - (n-1)^3 = 3n^2 - 3n + 1$. By Stolz–Cesàro:

$$\lim \frac{a_n}{b_n} = \lim \frac{n^2}{3n^2 - 3n + 1} = \frac{1}{3}.$$

(This is consistent with the formula $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$.)

Example 7.2. Compute $\lim_{n\to\infty} \frac{1^p+2^p+\cdots+n^p}{n^{p+1}}$ for $p > 0$.

Solution. With $a_n = \sum_{k=1}^{n} k^p$ and $b_n = n^{p+1}$:

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \frac{n^p}{n^{p+1} - (n-1)^{p+1}}.$$

Since $n^{p+1} - (n-1)^{p+1} \sim (p+1)n^p$ as $n \to \infty$, the limit is $\frac{1}{p+1}$.

7.9 The Ratio Test for Sequences

Theorem 7.10. Let $\langle a_n \rangle$ be a sequence of positive real numbers. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, then $\lim_{n\to\infty} a_n^{1/n} = L$.

Proof. This follows from Theorem 7.8 applied to the sequence $r_n = a_{n+1}/a_n$. We have $(r_1 r_2 \cdots r_{n-1})^{1/(n-1)} \to L$. Since $r_1 r_2 \cdots r_{n-1} = a_n/a_1$, we get $(a_n/a_1)^{1/(n-1)} \to L$, and then $a_n^{1/n} \to L$. □

7.10 Cesàro's Theorem on Products

Theorem 7.11 (Cesàro). If $\lim a_n = L$ and $\lim b_n = M$, then

$$\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} = LM.$$

7.11 Geometric Sequences

Theorem 7.12. For the geometric sequence $\langle r^n \rangle$:

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1, \\ 1 & \text{if } r = 1, \\ +\infty & \text{if } r > 1, \\ \text{does not exist} & \text{if } r \le -1. \end{cases}$$

Proof. Case $|r| < 1$: Write $|r| = \frac{1}{1+h}$ with $h > 0$. Then $|r^n| = |r|^n = \frac{1}{(1+h)^n} \le \frac{1}{1+nh} \to 0$ by Bernoulli's inequality.

Case $r = 1$: Trivial.

Case $r > 1$: Write $r = 1 + h$ with $h > 0$. Then $r^n = (1+h)^n \ge 1 + nh \to +\infty$.

Case $r = -1$: The sequence $\langle (-1)^n \rangle$ oscillates.

Case $r < -1$: $|r^n| = |r|^n \to +\infty$, but the signs alternate, so the sequence diverges by oscillation. □

7.12 Recursive Sequences and Fixed Points

Example 7.3. Define $a_1 = 1$ and $a_{n+1} = \frac{a_n + 3}{a_n + 1}$ for $n \ge 1$. Find $\lim a_n$.

Solution. First compute a few terms: $a_1 = 1$, $a_2 = 2$, $a_3 = 5/3 \approx 1.667$, $a_4 = 8/5 = 1.6$, $a_5 = 23/13 \approx 1.615$.

The terms appear to converge. If $L = \lim a_n$ exists, then taking limits:

$$L = \frac{L+3}{L+1} \implies L^2 + L = L + 3 \implies L^2 = 3 \implies L = \sqrt{3}.$$

(Taking the positive root since $a_n > 0$.)

To prove convergence rigorously, one can show that the odd-indexed subsequence is increasing and the even-indexed subsequence is decreasing, and both are bounded, then use the Monotone Convergence Theorem.

7.13 The Fibonacci Sequence and the Golden Ratio

Theorem 7.13. Let F_n denote the n-th Fibonacci number ($F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$). Then

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180.$$

Proof. Define $r_n = F_{n+1}/F_n$. From $F_{n+1} = F_n + F_{n-1}$:

$$r_n = \frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} = 1 + \frac{1}{r_{n-1}}.$$

We show the odd-indexed terms $r_1, r_3, r_5, \ldots$ form an increasing sequence bounded above, and the even-indexed terms $r_2, r_4, r_6, \ldots$ form a decreasing sequence bounded below. Both converge to the same limit L satisfying:

$$L = 1 + \frac{1}{L} \implies L^2 = L + 1 \implies L = \frac{1+\sqrt{5}}{2} = \varphi. \qquad \square$$

Remark 7.3. The golden ratio φ satisfies $\varphi^2 = \varphi + 1$, $1/\varphi = \varphi - 1$, and appears as the limit of ratios of consecutive terms in any generalised Fibonacci sequence (where $G_{n+1} = G_n + G_{n-1}$ with any positive initial conditions).

8 Series as Sequences of Partial Sums

8.1 From Sequences to Series

Definition 8.1 (Series). Given a sequence $\langle a_n \rangle$, the infinite series $\sum_{n=1}^{\infty} a_n$ is defined as the limit

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N, \qquad \text{where } S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N$$

is the N-th partial sum. The series converges if the sequence $\langle S_N \rangle$ converges; otherwise it diverges.

Remark 8.1. The study of series is entirely reducible to the study of the sequence of partial sums. Every theorem about sequences can be applied to investigate series convergence.

8.2 The Divergence Test

Theorem 8.1 (Divergence Test / n-th Term Test). If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. If $S_N \to S$, then $a_n = S_n - S_{n-1} \to S - S = 0$. $\square$

Remark 8.2. The converse is false: $\lim 1/n = 0$ but $\sum 1/n$ diverges (the harmonic series).

8.3 Geometric Series

Theorem 8.2. For $|r| < 1$: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. For $|r| \geq 1$, the geometric series diverges.

Proof. The partial sum is $S_N = \frac{1-r^{N+1}}{1-r}$. Since $r^{N+1} \to 0$ when $|r| < 1$, we get $S_N \to \frac{1}{1-r}$. $\square$

8.4 Telescoping Series

Example 8.1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

Solution. By partial fractions: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. The partial sum telescopes:

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{N+1} \to 1.$$

8.5 Cauchy Criterion for Series

Theorem 8.3. $\sum a_n$ converges if and only if for every $\varepsilon > 0$, there exists N such that

$$\left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon \qquad \text{for all } n > m \geq N.$$

This is simply the Cauchy criterion (Theorem 5.3) applied to the sequence of partial sums.

8.6 The Harmonic Series Diverges

Theorem 8.4. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof (Oresme, c. 1350). Group the terms:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \tag{3}$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = +\infty. \tag{4}$$

Each group of 2^{k-1} terms (from $n = 2^{k-1} + 1$ to $n = 2^k$) has sum $\geq 1/2$. □

Remark 8.3. Despite $1/n \to 0$, the harmonic series diverges. This shows that the converse of the Divergence Test is false, and illustrates why sequence convergence alone does not determine series convergence.

8.7 The p-Series

Theorem 8.5. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof sketch for $p > 1$. By the Cauchy condensation test, $\sum 1/n^p$ converges iff $\sum 2^n/(2^n)^p = \sum (2^{1-p})^n$ converges, which holds iff $2^{1-p} < 1$, i.e., $p > 1$. □

8.8 Preview of Convergence Tests

The following tests for series convergence can all be derived from properties of sequences:

- Comparison Test: If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- Ratio Test: If $\lim |a_{n+1}/a_n| = L < 1$, the series converges absolutely.
- Root Test: If $\lim |a_n|^{1/n} = L < 1$, the series converges absolutely.
- Alternating Series Test (Leibniz): If $a_n > 0$, a_n is decreasing, and $a_n \to 0$, then $\sum (-1)^{n+1} a_n$ converges.
- p-Series Test: $\sum 1/n^p$ converges iff $p > 1$.

9 Applications and Further Topics

9.1 Sequences in Metric Spaces

The theory of sequences generalises naturally to metric spaces.

Definition 9.1 (Metric Space). A metric space (X, d) consists of a set X and a function $d : X \times X \to [0, \infty)$ satisfying:

1. $d(x, y) = 0 \iff x = y$;
2. $d(x, y) = d(y, x)$ (symmetry);
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Definition 9.2. A sequence $\langle x_n \rangle$ in (X, d) converges to $x \in X$ if $d(x_n, x) \to 0$. It is Cauchy if for every $\varepsilon > 0$, there exists N with $d(x_n, x_m) < \varepsilon$ for $n, m \geq N$. The space is complete if every Cauchy sequence converges.

Example 9.1. $\mathbb{R}$ with $d(x, y) = |x - y|$ is a complete metric space (this is the Cauchy criterion, Theorem 5.3).

Example 9.2. $\mathbb{Q}$ with $d(x, y) = |x - y|$ is *not* complete. The sequence $1, 1.4, 1.41, 1.414, \ldots$ (approximations to $\sqrt{2}$) is Cauchy in $\mathbb{Q}$ but has no limit in $\mathbb{Q}$.

9.2 Contraction Mappings and Fixed Point Theorems

Definition 9.3 (Contraction Mapping). Let (X, d) be a metric space. A function $T : X \to X$ is a contraction if there exists $0 \leq c < 1$ such that

$$d(T(x), T(y)) \leq c \cdot d(x, y) \qquad \text{for all } x, y \in X.$$

Theorem 9.1 (Banach Fixed Point Theorem). Let (X, d) be a complete metric space and $T : X \to X$ a contraction with constant c. Then:

1. T has a unique fixed point $x^* \in X$ (i.e., $T(x^*) = x^*$).
2. For any $x_0 \in X$, the sequence $x_{n+1} = T(x_n)$ converges to x^*.
3. The rate of convergence satisfies $d(x_n, x^*) \leq \frac{c^n}{1-c} d(x_0, x_1)$.

Proof. Existence. Let $x_0 \in X$ and $x_{n+1} = T(x_n)$. Then:

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq c \cdot d(x_n, x_{n-1}) \leq \cdots \leq c^n \cdot d(x_1, x_0).$$

For $n > m$:

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) && (5) \\ &\leq (c^{n-1} + \cdots + c^m) d(x_1, x_0) && (6) \\ &\leq \frac{c^m}{1 - c} d(x_1, x_0). && (7) \end{aligned}$$

Since $c^m \to 0$, $\langle x_n \rangle$ is Cauchy. By completeness, $x_n \to x^*$ for some $x^* \in X$.

x^* is a fixed point. By continuity of T (contractions are Lipschitz, hence continuous):

$$T(x^*) = T(\lim x_n) = \lim T(x_n) = \lim x_{n+1} = x^*.$$

Uniqueness. If $T(y^*) = y^*$ also, then $d(x^*, y^*) = d(T(x^*), T(y^*)) \leq c \cdot d(x^*, y^*)$. Since $c < 1$, this forces $d(x^*, y^*) = 0$, i.e., $x^* = y^*$. □

Banach Fixed Point Theorem: Every contraction on a complete metric space has a unique fixed point, obtainable as the limit of the iteration $x_{n+1} = T(x_n)$ from any starting point.

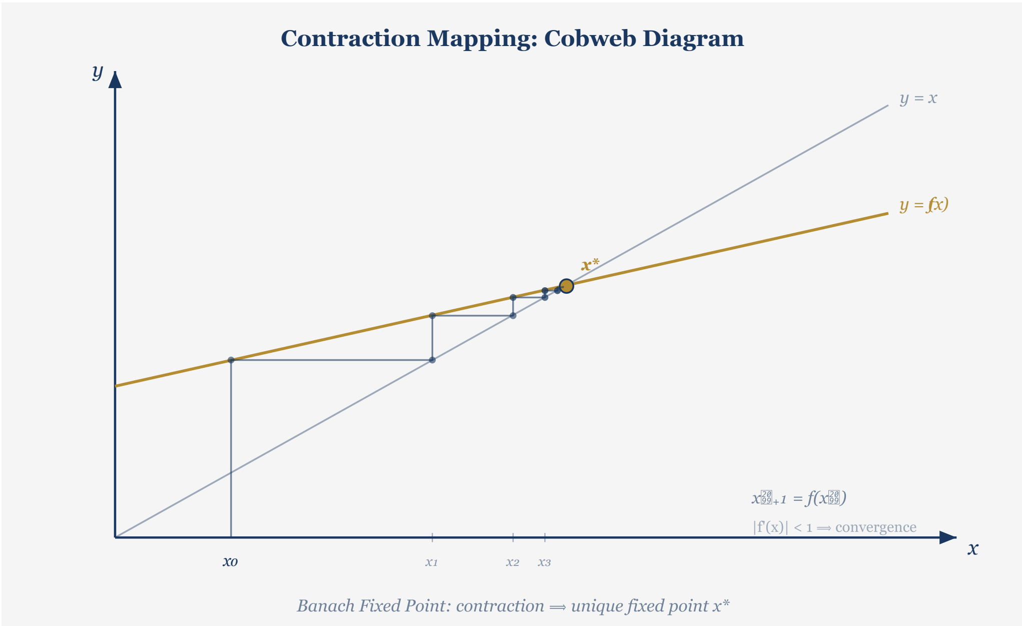

Figure 7: A contraction mapping iteratively applied: successive iterates $x_0, x_1, x_2, \ldots$ spiral toward the unique fixed point x^* where $T(x^*) = x^*$.

Example 9.3 (Application to the Babylonian method). The function $T(x) = \frac{1}{2}(x+c/x)$ for computing $\sqrt{c}$ (Example 4.4) is a contraction on any interval $[\sqrt{c}, M]$ for $M > \sqrt{c}$, with contraction constant $c = 1 - c/M^2 < 1$. The Banach theorem guarantees convergence to $\sqrt{c}$.

Example 9.4 (Solving equations by iteration). To solve $x = \cos x$, define $T(x) = \cos x$ on $[0, 1]$. Since $|T'(x)| = |\sin x| \le \sin 1 \approx 0.841 < 1$, T is a contraction. Starting from $x_0 = 0.5$:

$$x_1 = 0.8776, \quad x_2 = 0.6390, \quad x_3 = 0.8027, \quad \ldots \to x^* \approx 0.7391.$$

This is the unique solution to $x = \cos x$.

9.3 Sequences and Topology: Compactness

Definition 9.4. A subset K of $\mathbb{R}$ is sequentially compact if every sequence in K has a subsequence that converges to a point in K.

Theorem 9.2 (Heine–Borel, Sequential Version). A subset $K \subseteq \mathbb{R}$ is sequentially compact if and only if it is closed and bounded.

This is essentially the Bolzano–Weierstrass theorem combined with the fact that closed sets contain their limit points.

9.4 Sequences and Continuity

Theorem 9.3 (Sequential Characterisation of Continuity). A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0 if and only if for every sequence $\langle x_n \rangle$ with $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.

This theorem shows that sequential convergence completely determines the topology of $\mathbb{R}$, and hence all continuous phenomena.

9.5 Sequences and Differentiability

Theorem 9.4. If f is differentiable at x_0, then for any sequence $h_n \to 0$ with $h_n \neq 0$:

$$f'(x_0) = \lim_{n\to\infty} \frac{f(x_0 + h_n) - f(x_0)}{h_n}.$$

This sequential characterisation of the derivative is often used to prove that a function is *not* differentiable: if two different sequences $h_n \to 0$ give different limits for the difference quotient, then $f'(x_0)$ does not exist.

9.6 Picard Iteration and Differential Equations

The Banach Fixed Point Theorem underlies the Picard–Lindelöf theorem, which guarantees existence and uniqueness of solutions to ordinary differential equations.

Theorem 9.5 (Picard–Lindelöf, Sketch). Consider the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$, where f is Lipschitz in y. Define the Picard iterates:

$$y_0(t) = y_0, \qquad y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\, ds.$$

The sequence $\langle y_n(t)\rangle$ converges uniformly on a neighbourhood of t_0 to the unique solution $y(t)$.

Example 9.5. For $y' = y$, $y(0) = 1$, the Picard iterates are:

$$y_0(t) = 1, \tag{8}$$

$$y_1(t) = 1 + \int_0^t 1\, ds = 1 + t, \tag{9}$$

$$y_2(t) = 1 + \int_0^t (1 + s)\, ds = 1 + t + \frac{t^2}{2}, \tag{10}$$

$$y_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!} \to e^t. \tag{11}$$

The sequence of partial sums of the exponential series converges to the solution $y = e^t$.

9.7 Newton's Method as a Sequence

Newton's method for finding roots of $f(x) = 0$ produces a sequence:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Theorem 9.6 (Quadratic Convergence). If f is twice continuously differentiable, $f(r) = 0$, $f'(r) \neq 0$, and x_0 is sufficiently close to r, then the Newton sequence $\langle x_n\rangle$ converges to r, and

$$|x_{n+1} - r| \leq C|x_n - r|^2$$

for some constant $C > 0$. This is quadratic convergence—the number of correct digits roughly doubles at each step.

Example 9.6. To find $\sqrt{2}$ using Newton's method on $f(x) = x^2 - 2$:

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n + 2/x_n}{2}.$$

Starting from $x_0 = 1$: $x_1 = 1.5$, $x_2 = 1.41\overline{6}$, $x_3 = 1.41421568\ldots$, $x_4 = 1.41421356237\ldots$ Note that this is exactly the Babylonian method of Example 4.4.

10 Divergent and Oscillatory Sequences

For completeness, we classify sequences that do not converge.

10.1 Divergence to Infinity

Definition 10.1. $\langle a_n \rangle$ diverges to $+\infty$ if for every $M > 0$, there exists N such that $a_n > M$ for all $n \geq N$. We write $a_n \to +\infty$.

Similarly, $a_n \to -\infty$ if for every $M > 0$, there exists N with $a_n < -M$ for $n \geq N$.

Example 10.1. $\langle n^2 \rangle \to +\infty$. Given $M > 0$, choose $N > \sqrt{M}$. Then $n^2 > N^2 > M$ for $n \geq N$.

10.2 Oscillatory Sequences

Definition 10.2. A sequence oscillates finitely if it is bounded but non-convergent. It oscillates infinitely if it is unbounded but does not diverge to $+\infty$ or $-\infty$.

Example 10.2. $\langle (-1)^n \rangle$ oscillates finitely between -1 and 1.

Example 10.3. $\langle (-1)^n n \rangle = \langle -1, 2, -3, 4, \ldots \rangle$ oscillates infinitely.

Remark 10.1. Every real sequence falls into exactly one of four categories: convergent, divergent to $+\infty$, divergent to $-\infty$, or oscillatory.

Example 10.4 (A more subtle oscillatory sequence). The sequence $a_n = \sin(n)$ oscillates finitely. It is bounded ($|a_n| \leq 1$) but does not converge, because one can show that $\sin n$ is dense in $[-1, 1]$ (since π is irrational, the sequence $n \bmod 2\pi$ is equidistributed in $[0, 2\pi)$ by Weyl's theorem). Hence every point in $[-1, 1]$ is a limit point.

11 Additional Worked Examples

This section collects additional examples that illustrate the techniques developed throughout this document.

Example 11.1 (Limit of a ratio of factorials). Find $\lim_{n\to\infty} \frac{(2n)!}{4^n (n!)^2}$.

Solution. Let $a_n = \frac{(2n)!}{4^n (n!)^2}$. Compute the ratio:

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{4^{n+1}((n+1)!)^2} \cdot \frac{4^n (n!)^2}{(2n)!} = \frac{(2n+2)(2n+1)}{4(n+1)^2} = \frac{2(2n+1)}{4(n+1)} = \frac{2n+1}{2n+2}.$$

Since $\frac{a_{n+1}}{a_n} = \frac{2n+1}{2n+2} < 1$, the sequence is decreasing. It is bounded below by 0, so by the Monotone Convergence Theorem, it converges. By Stirling's approximation, $a_n \sim \frac{1}{\sqrt{\pi n}} \to 0$.

Example 11.2 (A sequence defined by integration). Let $a_n = \int_0^1 x^n e^x \, dx$ for $n \geq 0$. Show $a_n \to 0$.

Solution. For $x \in [0, 1]$: $0 \leq x^n e^x \leq e \cdot x^n$. Hence:

$$0 \leq a_n \leq e \int_0^1 x^n \, dx = \frac{e}{n+1} \to 0.$$

By the Squeeze Theorem, $a_n \to 0$.

Example 11.3 (Sequence involving n-th roots). Find $\lim_{n\to\infty} \sqrt[n]{3^n + 5^n}$.

Solution. Since $5^n \le 3^n + 5^n \le 2 \cdot 5^n$:

$$5 = \sqrt[n]{5^n} \le \sqrt[n]{3^n + 5^n} \le \sqrt[n]{2} \cdot 5.$$

Since $\sqrt[n]{2} \to 1$, the Squeeze Theorem gives $\lim = 5$.

Generalisation: For $a_1, \dots, a_k > 0$: $\lim_{n\to\infty}(a_1^n + \cdots + a_k^n)^{1/n} = \max(a_1, \dots, a_k)$.

Example 11.4 (A challenging recursive sequence). Let $a_1 = 2$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{3}{a_n})$. Show convergence and find the limit.

Solution. This is the Babylonian method for $\sqrt{3}$. By AM–GM: $a_{n+1} = \frac{1}{2}(a_n + 3/a_n) \ge \sqrt{3}$ for all $n \ge 1$. Also, for $n \ge 1$:

$$a_{n+1} - a_n = \frac{1}{2}\left(\frac{3}{a_n} - a_n\right) = \frac{3 - a_n^2}{2a_n} \le 0$$

since $a_n \ge \sqrt{3}$. So $\langle a_n \rangle_{n\ge1}$ is decreasing and bounded below by $\sqrt{3}$. By MCT, $L = \lim a_n$ exists and satisfies $L = \frac{1}{2}(L + 3/L)$, giving $L = \sqrt{3}$.

Example 11.5 (Toeplitz's Theorem application). Show that $\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{\ln n} = 1$.

Solution. Let $H_n = \sum_{k=1}^n 1/k$ denote the n-th harmonic number. It is known that $H_n = \ln n + \gamma + O(1/n)$, where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant. Therefore:

$$\frac{H_n}{\ln n} = 1 + \frac{\gamma}{\ln n} + O\left(\frac{1}{n \ln n}\right) \to 1.$$

Example 11.6 (Cesàro mean of a divergent sequence). The sequence $a_n = (-1)^{n+1}$ does not converge, but its Cesàro means are:

$$\sigma_n = \frac{a_1 + a_2 + \cdots + a_n}{n} = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

In both cases, $\sigma_n \to 0$. This shows that Cesàro summation can assign a value to some divergent sequences.

Example 11.7 (Stolz–Cesàro in action). Compute $\lim\limits_{n\to\infty} \dfrac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n\sqrt{n}}$.

Solution. Set $a_n = \sum_{k=1}^n \sqrt{k}$ and $b_n = n^{3/2}$. Then:

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \frac{\sqrt{n}}{n^{3/2} - (n-1)^{3/2}}.$$

For large n: $n^{3/2} - (n-1)^{3/2} = n^{3/2}\left(1 - (1 - 1/n)^{3/2}\right) \approx n^{3/2} \cdot \frac{3}{2n} = \frac{3}{2}\sqrt{n}$.

Hence $\frac{\sqrt{n}}{(3/2)\sqrt{n}} = \frac{2}{3}$. By Stolz–Cesàro, the limit is $\boxed{2/3}$.

Summary of Key Results

1. Definition of convergence: $a_n \to L$ iff $\forall \varepsilon > 0, \exists N$: $|a_n - L| < \varepsilon$ for $n \ge N$.
2. Uniqueness: Limits are unique.
3. Convergent $\Rightarrow$ Bounded: Every convergent sequence is bounded.

4. Algebra of Limits: Limits respect $+, -, \times, \div$.
5. Squeeze Theorem: Bounding from above and below determines the limit.
6. Monotone Convergence Theorem: Bounded monotone sequences converge.
7. Bolzano–Weierstrass: Every bounded sequence has a convergent subsequence.
8. Cauchy Criterion: In $\mathbb{R}$, convergence $\iff$ Cauchy.
9. lim sup / lim inf: Convergence $\iff \limsup = \liminf$.
10. Banach Fixed Point: Contractions on complete spaces have unique fixed points.

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