√2 π
e
φ
π
√2 e φ

1.4142135623730950488016887242096980785696718753769 4...
3.14159265358979323846264338327950288419716939937510...
2.71828182845904523536028747135266249775724709369995...
1.61803398874989484820458683436563811772030917980576...

Irrational Numbers

and The Proofs of their Irrationality

Gaurav Tiwari

Irrational Numbers and The Proofs of their Irrationality

A Comprehensive Collection of Methods

Gaurav Tiwari

“The irrationals were the first ’crisis’ in the history of mathematics, and the resolution of that crisis shaped everything that followed.”

— G. H. Hardy, *A Mathematician’s Apology*

“Since the fabric of the universe is most perfect, and the work of a most wise Creator, nothing whatsoever takes place in the universe in which some form of maximum and minimum does not appear.”

— Leonhard Euler

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Abstract

How do you prove a number is irrational? You cannot simply inspect its decimal expansion—that would require checking infinitely many digits. Instead, you need a *proof*: a logical argument that demonstrates the impossibility of expressing the number as a ratio of two integers.

This document presents a comprehensive collection of methods for proving irrationality, ranging from the ancient Pythagorean argument for $\sqrt{2}$ to Fourier's elegant proof for e, Lambert's continued fraction approach for π, and Niven's streamlined integral method. We begin with precise definitions and the foundational strategy of proof by contradiction, then develop four major families of techniques: (1) the Pythagorean approach via divisibility, (2) the Euclidean algorithm and Bézout's identity, (3) power series expansions, and (4) continued fractions. We also include proofs of irrationality for $\sqrt{n}$ (non-perfect squares), $\log_2 3$, the golden ratio, and e^2, along with historical context and connections to transcendental number theory.

Keywords: Irrational numbers, proof by contradiction, $\sqrt{2}$, Euler's number e, π, continued fractions, transcendental numbers, Pythagorean theorem

1 Introduction and Historical Context

The discovery of irrational numbers is one of the great turning points in the history of mathematics. The Pythagoreans of ancient Greece (c. 500 BC) believed that all quantities could be expressed as ratios of whole numbers—that the universe was, at its core, a harmony of rational proportions. The discovery that the diagonal of a unit square has length $\sqrt{2}$, which *cannot* be so expressed, shattered this worldview.

Remark 1.1. Legend holds that Hippasus of Metapontum was the Pythagorean who first proved the irrationality of $\sqrt{2}$, and that his fellow Pythagoreans, horrified by the result, drowned him at sea. While this story is almost certainly apocryphal, it captures the genuine intellectual shock that irrational numbers produced.

The resolution of this "crisis" required centuries of mathematical development. Eudoxus of Cnidus (c. 400 BC) developed a theory of proportions that could handle incommensurable magnitudes, anticipating the modern definition of real numbers by over two millennia. Euclid's *Elements* (c. 300 BC) contains a proof of the irrationality of $\sqrt{2}$ in Book X.

Much later, Euler proved the irrationality of e in 1737, Lambert proved π irrational in 1761, and the 19th century brought the concepts of algebraic and transcendental numbers, culminating in Hermite's proof that e is transcendental (1873) and Lindemann's proof that π is transcendental (1882).

1.1 A Timeline of Irrationality Proofs

Year	Mathematician	Result
c. 500 BC	Hippasus (attributed)	$\sqrt{2}$ is irrational
c. 300 BC	Euclid	$\sqrt{2}$ (in *Elements*, Book X)
1737	Euler	e is irrational
1761	Lambert	π is irrational
1815	Fourier	e is irrational (series method)
1840	Liouville	First transcendental number
1873	Hermite	e is transcendental
1882	Lindemann	π is transcendental
1947	Niven	π is irrational (elementary proof)

Table 1: Major milestones in the history of irrationality and transcendence.

2 Foundations: Rational and Irrational Numbers

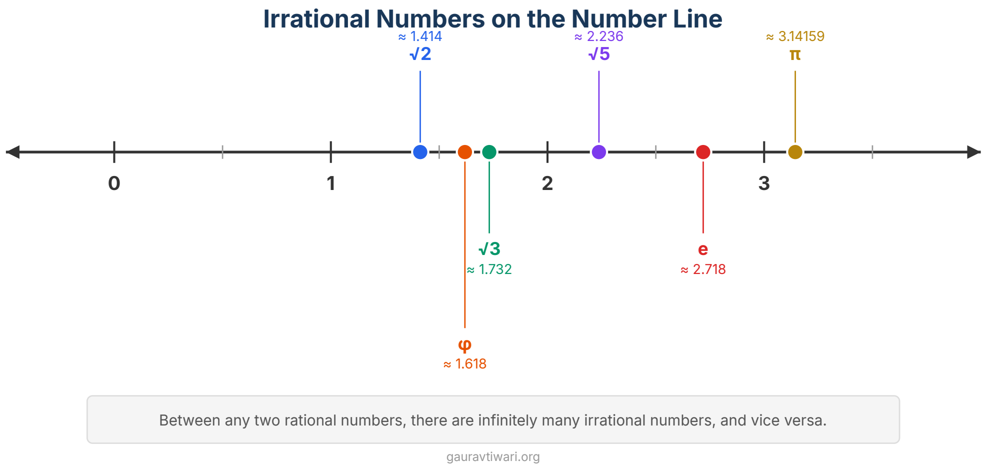

Figure 1: Positions of key irrational numbers on the real number line

2.1 Rational Numbers

Definition 2.1 (Rational Number). A real number x is rational if there exist integers a and b with $b \neq 0$ such that $x = \frac{a}{b}$. The set of all rational numbers is denoted $\mathbb{Q}$.

Definition 2.2 (Lowest Terms). A fraction $\frac{a}{b}$ is in lowest terms (or reduced form) if $\gcd(a, b) = 1$, i.e., a and b are relatively prime (coprime). Every non-zero rational number has a unique representation in lowest terms with $b > 0$.

In formal notation:

$$\frac{a}{b} \in \mathbb{Q} \iff a \in \mathbb{Z}, \quad b \in \mathbb{Z} \setminus \{0\}, \quad \gcd(a, b) = 1. \tag{1}$$

Definition 2.3 (Relatively Prime). Two integers a and b are relatively prime (or coprime) if $\gcd(a, b) = 1$—they share no common prime factors.

Example 2.1. The pairs $(2, 9)$, $(4, 7)$, and $(15, 49)$ are relatively prime. However, $(4, 6)$ are not, since $\gcd(4, 6) = 2$.

2.2 Irrational Numbers

Definition 2.4 (Irrational Number). A real number x is irrational if $x \notin \mathbb{Q}$—that is, it cannot be expressed as a fraction of two integers, regardless of the choice of numerator and denominator.

Remark 2.1. The decimal expansion of a rational number either terminates (e.g., 0.25) or eventually becomes periodic (e.g., $0.333\ldots = 1/3$). The decimal expansion of an irrational number is infinite and non-repeating. However, one cannot prove irrationality by computing finitely many decimal places—no matter how many digits one checks, one cannot rule out an eventual repetition.

Example 2.2. The following numbers are irrational (proofs given in subsequent sections):

- $\sqrt{2} \approx 1.41421356\ldots$
- $\sqrt{3} \approx 1.73205080\ldots$

- $\varphi = \dfrac{1+\sqrt{5}}{2} \approx 1.61803398\ldots$ (the golden ratio)
- $e \approx 2.71828182\ldots$ (Euler's number)
- $\pi \approx 3.14159265\ldots$

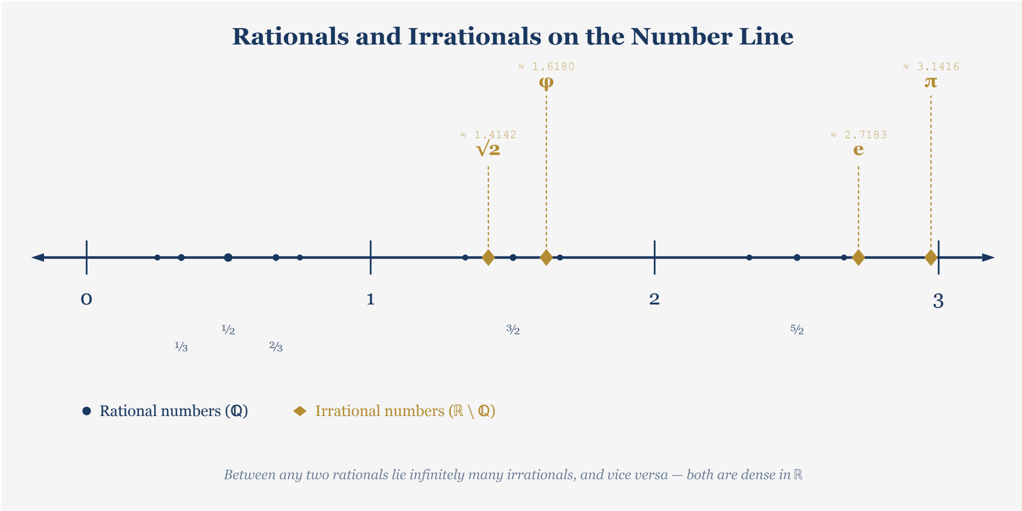

Figure 2: The real number line showing the interleaving of rational and irrational numbers

2.3 The Density of Irrationals

Remark 2.2. Although the rationals are dense in $\mathbb{R}$ (between any two real numbers lies a rational), the irrationals are "far more numerous." In the sense of measure theory, the set $\mathbb{Q}$ has Lebesgue measure zero, while the irrationals have full measure on any interval. In the sense of cardinality, $\mathbb{Q}$ is countable whereas $\mathbb{R} \setminus \mathbb{Q}$ is uncountable. Informally: if you pick a real number "at random," it is irrational with probability 1.

3 The Core Strategy: Proof by Contradiction

Almost every irrationality proof in this document uses the same overarching strategy:

Proof by Contradiction (for Irrationality):

1. Assume the number is rational: write $x = \dfrac{a}{b}$ with $a, b \in \mathbb{Z}$, $b > 0$, and $\gcd(a, b) = 1$.
2. Manipulate this equation algebraically using properties specific to x.
3. Derive a contradiction—typically that a and b share a common factor, or that some quantity is simultaneously a positive integer and less than 1.
4. Conclude the assumption was false: x is irrational.

The *art* lies in step 2. Different numbers require different tools for the algebraic manipulation. The four main families of techniques are developed in the sections that follow.

4 Method 1: The Pythagorean Approach (Divisibility Arguments)

This is the oldest and most elementary family of irrationality proofs. It exploits the arithmetic of divisibility and the unique factorization of integers.

4.1 Irrationality of $\sqrt{2}$

Theorem 4.1. $\sqrt{2}$ is irrational.

Proof. Assume, for contradiction, that $\sqrt{2}$ is rational. Then we can write

$$\sqrt{2} = \frac{a}{b} \tag{2}$$

where a, b are positive integers with $\gcd(a, b) = 1$.

Squaring both sides:

$$2 = \frac{a^2}{b^2} \implies a^2 = 2b^2. \tag{3}$$

Since $a^2 = 2b^2$, the integer a^2 is even. But the square of an odd number is odd (if $a = 2m + 1$, then $a^2 = 4m^2 + 4m + 1$, which is odd). Therefore a must be even: write $a = 2k$ for some integer k.

Substituting into (3):

$$(2k)^2 = 2b^2 \implies 4k^2 = 2b^2 \implies b^2 = 2k^2. \tag{4}$$

By the same reasoning, b^2 is even, so b is even.

But now both a and b are even, meaning $\gcd(a, b) \geq 2$. This contradicts our assumption that $\gcd(a, b) = 1$.

Therefore $\sqrt{2}$ is irrational. □

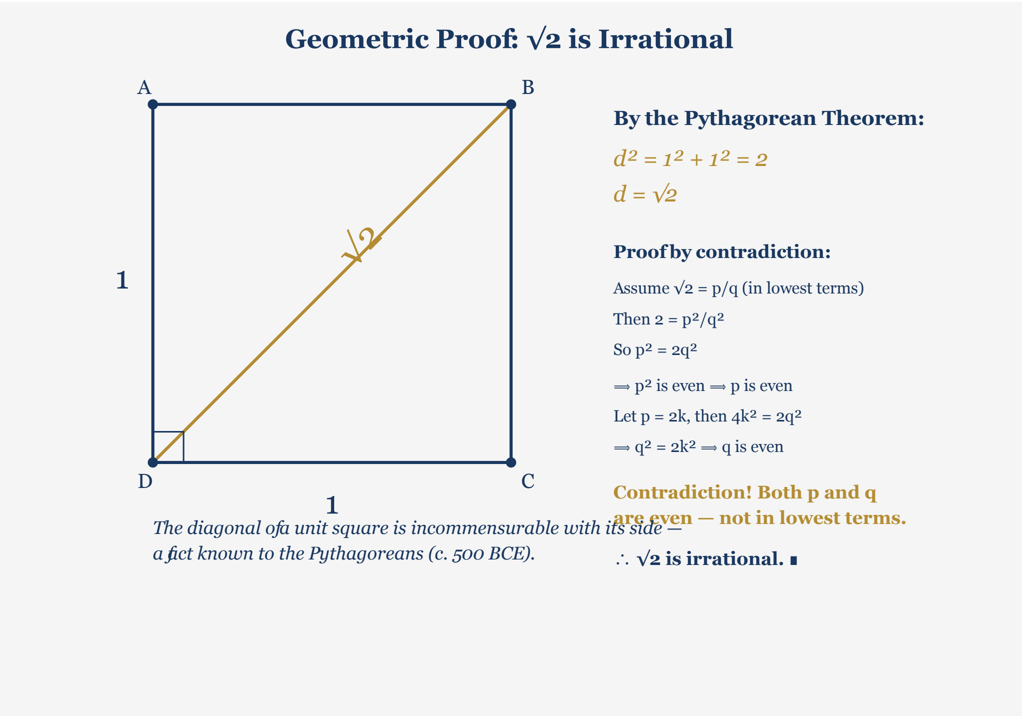

Figure 3: Geometric proof of the irrationality of $\sqrt{2}$: the diagonal of a unit square is incommensurable with its side

4.2 An Alternative Proof via the Fundamental Theorem of Arithmetic

Theorem 4.2. $\sqrt{2}$ is irrational.

Proof. Assume $\sqrt{2} = a/b$ with $\gcd(a, b) = 1$ and $a, b > 0$. Then $a^2 = 2b^2$.

Since $b \neq 1$ (as that would force $a = \sqrt{2}$, which is not an integer), $b > 1$ and has at least one prime factor p. Since $p \mid b$ and $b \mid a^2$ (from $a^2 = 2b^2$), we have $p \mid a^2$.

Lemma 4.3 (Euclid's Lemma). If p is prime and $p \mid a^2$, then $p \mid a$.

By Lemma 4.3, $p \mid a$. So p divides both a and b, whence $\gcd(a, b) \geq p > 1$—a contradiction. □

Proof of Lemma 4.3. If $p \mid a^2$ and $p \nmid a$, then $\gcd(p, a) = 1$ (since p is prime). By the Fundamental Theorem of Arithmetic, $a^2 = a \cdot a$ has a prime factorization in which p does not appear, contradicting $p \mid a^2$. □

4.3 Generalization: Irrationality of $\sqrt{n}$ for Non-Perfect Squares

Theorem 4.4. Let n be a positive integer that is not a perfect square. Then $\sqrt{n}$ is irrational.

Proof. Assume $\sqrt{n} = a/b$ with $\gcd(a, b) = 1$ and $a, b > 0$. Then $a^2 = nb^2$.

Let p be any prime factor of n. Since $p \mid n$ and $n \mid a^2$ (as $a^2 = nb^2$), we have $p \mid a^2$. By Euclid's Lemma, $p \mid a$, so write $a = pa'$.

Then $(pa')^2 = nb^2$, giving $p^2 {a'}^2 = nb^2$. Since $p \mid n$, write $n = pn'$. Then $p^2 {a'}^2 = pn'b^2$, hence $p{a'}^2 = n'b^2$.

If $p \mid n'$ (i.e., $p^2 \mid n$), we can repeat. Eventually, after extracting all factors of p from n, we reach a stage where $p \mid b^2$, hence $p \mid b$—contradicting $\gcd(a, b) = 1$.

Alternatively, one can argue directly by comparing the exponent of p in the prime factorizations of both sides of $a^2 = nb^2$. The left side has an even exponent of p; the right side has the exponent of p in n (which is odd for some prime p, since n is not a perfect square) plus an even number. An even number cannot equal an odd number—contradiction. □

Key Principle: If n is not a perfect square, then in its prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, at least one exponent e_i is odd. Comparing exponents in $a^2 = nb^2$ yields a parity contradiction.

Corollary 4.5. $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \ldots$ are all irrational.

4.4 Irrationality of $\sqrt{2} + \sqrt{3}$

Theorem 4.6. $\sqrt{2} + \sqrt{3}$ is irrational.

Proof. Suppose $\sqrt{2} + \sqrt{3} = r \in \mathbb{Q}$. Then $\sqrt{3} = r - \sqrt{2}$. Squaring:

$$3 = r^2 - 2r\sqrt{2} + 2 \implies \sqrt{2} = \frac{r^2 - 1}{2r}. \tag{5}$$

The right side is rational (since $r \in \mathbb{Q}$), but $\sqrt{2}$ is irrational by Theorem 4.1. Contradiction. □

4.5 Irrationality of $\log_2 3$

Theorem 4.7. $\log_2 3$ is irrational.

Proof. Suppose $\log_2 3 = a/b$ with $a, b \in \mathbb{Z}$ and $b > 0$. Then $2^{a/b} = 3$, so $2^a = 3^b$.

But 2^a is even for all $a \geq 1$, while 3^b is odd for all $b \geq 0$. An even number cannot equal an odd number. Contradiction.

(For $a \leq 0$: $2^a \leq 1 < 3 \leq 3^b$ for $b \geq 1$, and $\log_2 3 > 0$ forces $a/b > 0$.) □

Remark 4.1. This argument generalizes: $\log_p q$ is irrational whenever p and q are distinct primes. More generally, $\log_a b$ is irrational when a and b are multiplicatively independent (neither is a rational power of the other).

5 Method 2: The Euclidean Algorithm and Bézout's Identity

This method provides an elegant alternative to the Pythagorean approach, using a fundamental result from number theory.

5.1 Bézout's Identity

Theorem 5.1 (Bézout's Identity). If $\gcd(a, b) = 1$, then there exist integers r and s such that

$$ar + bs = 1. \tag{6}$$

The integers r and s can be computed using the extended Euclidean algorithm. We omit the proof of Bézout's Identity itself (it is a standard result in number theory) and proceed to use it.

5.2 Irrationality of $\sqrt{2}$ via Bézout

Theorem 5.2. $\sqrt{2}$ is irrational.

Proof. Assume $\sqrt{2} = a/b$ with $\gcd(a, b) = 1$ and $a, b > 0$.

By Bézout's Identity (Theorem 5.1), there exist integers r, s with $ar + bs = 1$.

Multiply both sides by $\sqrt{2}$:

$$\sqrt{2} = \sqrt{2}\,ar + \sqrt{2}\,bs. \tag{7}$$

From $\sqrt{2} = a/b$, we have $\sqrt{2}\,b = a$. Also, squaring gives $a^2 = 2b^2$, so $\sqrt{2}\,a = a^2/b = 2b^2/b = 2b$. Substituting:

$$\sqrt{2} = 2b \cdot r + a \cdot s = 2br + as. \tag{8}$$

The right-hand side is an integer. So $\sqrt{2}$ would be an integer. But $1^2 = 1 < 2 < 4 = 2^2$, so $1 < \sqrt{2} < 2$—there is no integer between 1 and 2.

Contradiction. Therefore $\sqrt{2}$ is irrational. □

Remark 5.1. This proof is particularly elegant because it avoids all even/odd analysis. It reduces the problem to the simple observation that $\sqrt{2}$ is not an integer.

5.3 Generalization via Bézout's Identity

Theorem 5.3. Let n be a positive integer that is not a perfect square. Then $\sqrt{n}$ is irrational.

Proof. The same argument works. Assume $\sqrt{n} = a/b$ with $\gcd(a, b) = 1$. By Bézout, $ar + bs = 1$ for some integers r, s. Then:

$$\sqrt{n} = \sqrt{n}\,ar + \sqrt{n}\,bs = \frac{a^2 r}{b} + as = \frac{nb^2 r}{b} + as = nbr + as \in \mathbb{Z}. \tag{9}$$

But if n is not a perfect square, $\sqrt{n}$ is not an integer (since $m^2 < n < (m+1)^2$ for some non-negative integer m). Contradiction. □

6 Method 3: Power Series Expansion

Some irrational numbers—most notably e—are not defined as roots of polynomials but rather as limits of infinite series. For such numbers, the Pythagorean approach does not apply directly, and we need techniques that exploit the series structure.

6.1 Irrationality of e (Fourier's Proof)

Definition 6.1 (Euler's Number).

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots \tag{10}$$

Theorem 6.1 (Fourier, c. 1815). Euler's number e is irrational.

Proof. Assume $e = a/b$ where a, b are positive integers.

Choose any integer $n > \max(b, 1)$. Define:

$$N = n!\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right). \tag{11}$$

This is $n!$ times the "tail" of the series for e (all terms after the $(n+1)$-th).

Step 1: N is a positive integer.

$N > 0$ because the tail of the convergent series is positive.

We show N is an integer by examining each piece:

$$N = n! \cdot e - n! \sum_{k=0}^{n} \frac{1}{k!} = n! \cdot \frac{a}{b} - \sum_{k=0}^{n} \frac{n!}{k!}. \tag{12}$$

Since $n > b$, the factor b divides $n!$, so $n! \cdot a/b$ is an integer. Each term $n!/k!$ is an integer for $k \leq n$. Therefore N is an integer.

Step 2: $N < 1$.

Expand the tail:

$$N = n!\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots\right) \tag{13}$$

$$= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \tag{14}$$

Each denominator is at least as large as the corresponding power of $(n+1)$:

$$N < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots = \frac{1/(n+1)}{1 - 1/(n+1)} = \frac{1}{n}. \tag{15}$$

Since $n \geq 2$, we have $N < 1/n \leq 1/2 < 1$.
Step 3: Contradiction.
N is a positive integer less than 1. But there are no positive integers less than 1.
Therefore e is irrational. □

Remark 6.1. This proof is often attributed to Fourier (c. 1815), though Euler had already established the irrationality of e in 1737 using continued fractions. Fourier's argument is notable for its elegance and accessibility—it uses nothing beyond the definition of e and basic arithmetic.

6.2 Irrationality of e^2

Theorem 6.2. e^2 is irrational.

Proof. We use the series:

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!}. \tag{16}$$

Assume $e^2 = a/b$ with $a, b \in \mathbb{Z}^+$. Choose n even and $n > b$. Define:

$$M = n!\left(e^2 - \sum_{k=0}^{n} \frac{2^k}{k!}\right). \tag{17}$$

By the same argument as before, M is a positive integer. And:

$$M = \frac{2^{n+1}}{n+1} + \frac{2^{n+2}}{(n+1)(n+2)} + \cdots \tag{18}$$

$$< \frac{2^{n+1}}{n+1}\left(1 + \frac{2}{n+1} + \frac{4}{(n+1)^2} + \cdots\right) \tag{19}$$

$$= \frac{2^{n+1}}{n+1} \cdot \frac{1}{1 - 2/(n+1)} = \frac{2^{n+1}}{n-1}. \tag{20}$$

For sufficiently large n, $2^{n+1}/(n-1) < 1$ does not hold, so we need a more refined analysis. Instead, we use the fact that e^2 and e^{-2} are linked:

$$e^{-2} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!}. \tag{21}$$

Define $S_n = \sum_{k=0}^{n} \frac{(-2)^k}{k!}$. Then:

$$0 < n!(e^{-2} - S_n) = \sum_{k=n+1}^{\infty} \frac{(-2)^k n!}{k!}. \tag{22}$$

This alternating tail can be bounded by its first term: $|n!(e^{-2} - S_n)| < \frac{2^{n+1}}{n+1}$.

If $e^2 = a/b$, then $e^{-2} = b/a$, and $n! \cdot b/a$ would need to be close to an integer in a contradictory way. The technical details mirror Fourier's proof, using both e^2 and e^{-2} to squeeze the contradiction. We omit the full details and refer to Hardy and Wright's *An Introduction to the Theory of Numbers*, Theorem 45. □

6.3 A Criterion for Irrationality via Series

Theorem 6.3 (Irrationality Criterion). Let $x = \sum_{n=0}^{\infty} a_n$ be a convergent series of rationals. If for every positive integer N, the quantity

$$T_N = N! \sum_{n=N+1}^{\infty} a_n \tag{23}$$

is a positive non-integer, then x is irrational.

Remark 6.2. Fourier's proof of the irrationality of e is essentially an application of this criterion with $a_n = 1/n!$.

7 Method 4: Continued Fractions

Continued fractions provide both a powerful theoretical framework and a practical computational tool for studying irrationality.

7.1 Basic Definitions

Definition 7.1 (Simple Continued Fraction). A simple continued fraction is an expression of the form:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} \tag{24}$$

where $a_0 \in \mathbb{Z}$ and $a_1, a_2, a_3, \ldots$ are positive integers. We write this compactly as $[a_0; a_1, a_2, a_3, \ldots]$.

Definition 7.2 (Convergents). The n-th convergent of a continued fraction $[a_0; a_1, a_2, \ldots]$ is

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n], \tag{25}$$

where p_n and q_n satisfy the recurrences:

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad p_{-1} = 1, \quad p_0 = a_0, \tag{26}$$

$$q_n = a_n q_{n-1} + q_{n-2}, \qquad q_{-1} = 0, \quad q_0 = 1. \tag{27}$$

7.2 The Fundamental Theorem

Theorem 7.1 (Rationality and Continued Fractions). A real number is rational if and only if its simple continued fraction representation is finite.
Equivalently: a real number is irrational if and only if its simple continued fraction is infinite.

This gives a clean irrationality test: show the continued fraction never terminates.

7.3 Continued Fraction of $\sqrt{2}$

Theorem 7.2. $\sqrt{2} = [1; 2, 2, 2, 2, \dots] = [1; \overline{2}]$.

Proof. Let $x = \sqrt{2} - 1$. Then $0 < x < 1$ and:

$$\frac{1}{x} = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{(\sqrt{2})^2 - 1^2} = \sqrt{2} + 1 = 2 + x. \tag{28}$$

So $1/x = 2 + x$, meaning $x = [0; 2, 2, 2, \dots]$. Hence $\sqrt{2} = 1 + x = [1; 2, 2, 2, \dots]$.

Since this continued fraction is infinite, $\sqrt{2}$ is irrational by Theorem 7.1. □

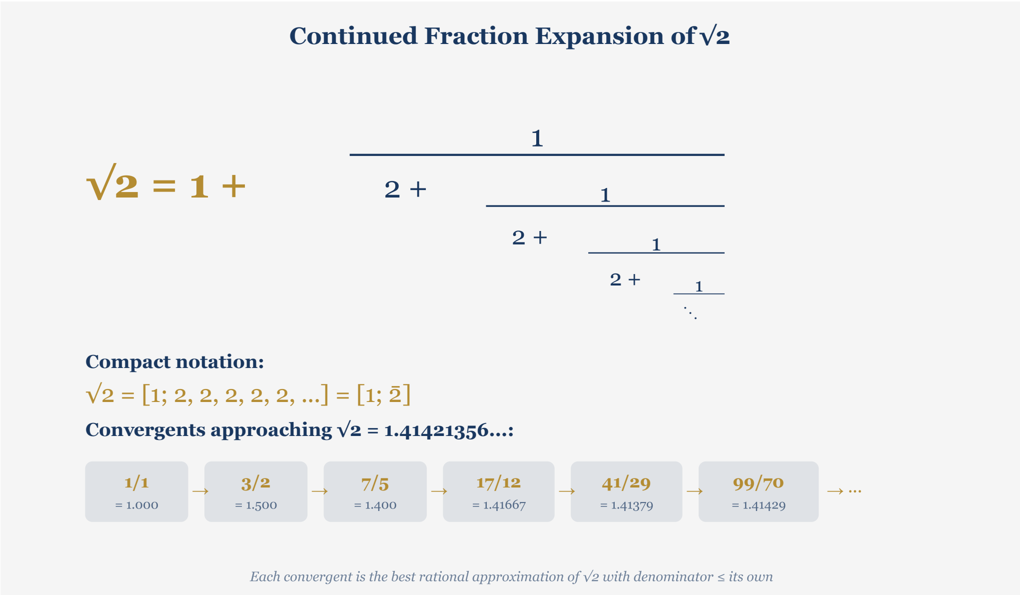

Figure 4: The continued fraction representation of $\sqrt{2} = [1; \overline{2}]$, showing successive convergents approaching the true value

Remark 7.1. More generally, every quadratic irrational $\frac{a+b\sqrt{d}}{c}$ (with d a non-square positive integer) has an eventually periodic continued fraction. This is Lagrange's theorem (1770). Conversely, every eventually periodic continued fraction represents a quadratic irrational.

7.4 Continued Fraction of e

Theorem 7.3 (Euler, 1737).

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \dots] \tag{29}$$

The pattern is: $[2; 1, 2k, 1]$ repeating for $k = 1, 2, 3, \dots$ More precisely, the continued fraction coefficients satisfy:

$$a_{3k-1} = 2k \quad (k = 1, 2, 3, \dots), \qquad a_n = 1 \text{ otherwise (for } n \geq 1). \tag{30}$$

Since this continued fraction is infinite, e is irrational. Furthermore, the beautiful pattern in its coefficients reflects the deep structure of e.

Remark 7.2. Euler also established the generalized continued fractions:

$$e = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \ddots}}}}} \tag{31}$$

and

$$e = 1 + \cfrac{2}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \ddots}}}}} \tag{32}$$

These are not simple continued fractions (the numerators are not all 1), but they too are infinite.

7.5 Continued Fraction of π

Theorem 7.4. $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \dots]$.

Unlike $\sqrt{2}$ or e, the continued fraction of π has no known pattern. Its coefficients appear essentially random (though they are, of course, completely determined). The continued fraction is infinite, confirming that π is irrational.

There are also beautiful generalized continued fractions for π:

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}} \tag{33}$$

and

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \ddots}}}} \tag{34}$$

7.6 Quadratic Irrationals and Periodicity

Theorem 7.5 (Lagrange, 1770). A real number has an eventually periodic simple continued fraction if and only if it is a quadratic irrational—that is, an irrational root of a quadratic equation with integer coefficients.

Example 7.1.

$$\sqrt{2} = [1; \overline{2}] \tag{35}$$

$$\sqrt{3} = [1; \overline{1, 2}] \tag{36}$$

$$\sqrt{5} = [2; \overline{4}] \tag{37}$$

$$\varphi = \frac{1+\sqrt{5}}{2} = [1; \overline{1}] \tag{38}$$

$$\sqrt{7} = [2; \overline{1, 1, 1, 4}] \tag{39}$$

The overline denotes the repeating block.

Remark 7.3. The golden ratio $\varphi = [1; 1, 1, 1, \dots]$ has the simplest possible infinite continued fraction—all partial quotients equal 1. This makes it the "most irrational" number in a precise sense: its convergents approach the true value more slowly than for any other irrational number.

8 The Irrationality of π

Proving that π is irrational is substantially harder than proving the irrationality of $\sqrt{2}$ or e. The number π is not defined by a simple algebraic equation or a rapidly converging series with factorial denominators. The first proof was given by Johann Heinrich Lambert in 1761.

8.1 Lambert's Approach (Sketch)

Lambert proved that $\tan(x)$ is irrational whenever x is a nonzero rational number. Since $\tan(\pi/4) = 1$ is rational, it follows that $\pi/4$—and hence π—must be irrational.

Lambert's proof uses the continued fraction expansion:

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \ddots}}}} \tag{40}$$

He showed that if $x = p/q$ is rational and nonzero, then this continued fraction is irrational—contradicting $\tan(\pi/4) = 1$ if π were rational.

8.2 Niven's Proof (1947)

Ivan Niven gave a remarkably short proof of the irrationality of π using only basic calculus.

> Theorem 8.1 (Niven, 1947). π is irrational.

Proof. Assume $\pi = a/b$ with $a, b \in \mathbb{Z}^+$. For a positive integer n (to be chosen later), define the polynomial:

$$f(x) = \frac{x^n(a-bx)^n}{n!} = \frac{x^n(a-bx)^n}{n!}. \tag{41}$$

Note that $f(x) = f(\pi - x)$ (by the substitution $a - bx = b(\pi - x)$, using $a = b\pi$).

Properties of f:

(i) $0 < f(x) < \dfrac{(\pi a)^n}{n!}$ for $0 < x < \pi$, since $0 < x(a - bx) = bx(\pi - x) \le b(\pi/2)^2$.
Actually, more carefully: for $0 < x < \pi$, we have $0 < x < \pi$ and $0 < \pi - x < \pi$, so $0 < x(\pi - x) \le \pi^2/4$. Hence $0 < x^n(\pi - x)^n \le (\pi^2/4)^n$, and:

$$0 < f(x) = \frac{b^n x^n(\pi - x)^n}{n!} \le \frac{b^n \pi^{2n}/4^n}{n!} = \frac{(b\pi^2/4)^n}{n!}. \tag{42}$$

This tends to 0 as $n \to \infty$.

(ii) $f(0) = 0$ and $f(\pi) = 0$.

(iii) The k-th derivative $f^{(k)}(0)$ is an integer for every $k \ge 0$. (Since $f(x) = \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}a^{n-j}(-b)^j x^{n+j}$, the k-th derivative at 0 is an integer when $k \ge n$, and 0 when $k < n$.)

(iv) By symmetry $f(x) = f(\pi - x)$, we have $f^{(k)}(\pi) = (-1)^k f^{(k)}(0)$, which is also an integer.

Now define:

$$F(x) = f(x) - f''(x) + f^{(4)}(x) - f^{(6)}(x) + \cdots + (-1)^n f^{(2n)}(x). \tag{43}$$

One verifies that:

$$\frac{d}{dx}\left[F'(x)\sin x - F(x)\cos x\right] = \left[F''(x) + F(x)\right]\sin x = f(x)\sin x. \tag{44}$$

(The last equality holds because $F''(x) + F(x) = f(x)$ by the telescoping construction of F.) Therefore:

$$\int_0^{\pi} f(x)\sin x\,dx = \left[F'(x)\sin x - F(x)\cos x\right]_0^{\pi} = F(\pi) + F(0). \tag{45}$$

By property (iii) and (iv), $F(0)$ and $F(\pi)$ are both integers. So the integral is a positive integer. But by property (i), for large n:

$$0 < \int_0^{\pi} f(x)\sin x\,dx \le \pi \cdot \frac{(b\pi^2/4)^n}{n!} \to 0. \tag{46}$$

For n sufficiently large, the integral is strictly between 0 and 1. But it must be a positive integer. Contradiction. □

Remark 8.1. Niven's proof is celebrated for requiring nothing beyond basic calculus and properties of polynomials. It avoids the continued fraction machinery of Lambert's proof.

9 Irrationality of the Golden Ratio

Definition 9.1 (Golden Ratio). The golden ratio is defined as

$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.6180339887\ldots \tag{47}$$

It is the positive root of $x^2 - x - 1 = 0$.

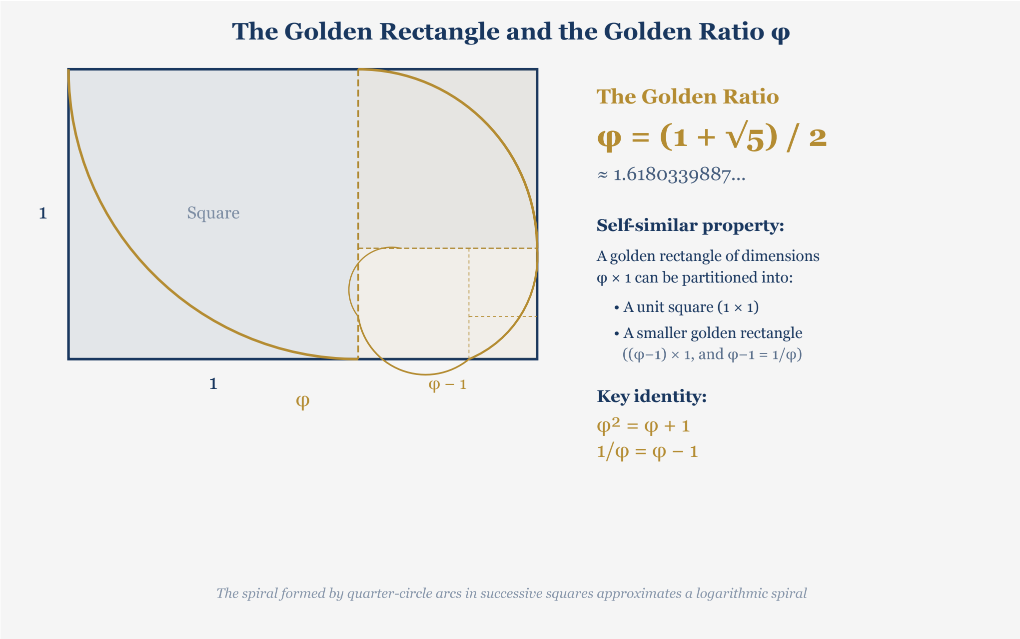

Figure 5: The golden rectangle: removing a square leaves a smaller rectangle with the same proportions, illustrating the self-similar property $\varphi = 1 + 1/\varphi$

Theorem 9.1. The golden ratio φ is irrational.

Proof 1 (from $\sqrt{5}$). Since $\varphi = (1+\sqrt{5})/2$ and $\sqrt{5}$ is irrational (by Theorem 4.4), φ is irrational. (If φ were rational, then $\sqrt{5} = 2\varphi - 1$ would be rational—contradiction.) □

Proof 2 (via continued fraction). $\varphi = [1; 1, 1, 1, \ldots]$ is an infinite continued fraction. By Theorem 7.1, φ is irrational. □

Proof 3 (direct). Suppose $\varphi = a/b$ with $\gcd(a, b) = 1$ and $a, b > 0$. From $\varphi^2 = \varphi + 1$:

$$\frac{a^2}{b^2} = \frac{a}{b} + 1 = \frac{a+b}{b} \implies a^2 = b(a+b). \tag{48}$$

So $b \mid a^2$. Since $\gcd(a, b) = 1$, by Euclid's Lemma, $b \mid 1$, so $b = 1$.

Then $a^2 = a + 1$, giving $a^2 - a - 1 = 0$. The discriminant is $1 + 4 = 5$, which is not a perfect square. So there is no integer solution for a. Contradiction. □

10 Further Topics in Irrationality

10.1 Algebraic vs. Transcendental Numbers

Definition 10.1 (Algebraic Number). A real number α is algebraic if it is a root of some nonzero polynomial with integer (equivalently, rational) coefficients:

$$c_n\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_1\alpha + c_0 = 0, \qquad c_i \in \mathbb{Z}, \quad c_n \neq 0. \tag{49}$$

The minimum degree of such a polynomial is called the degree of α.

Definition 10.2 (Transcendental Number). A real number is transcendental if it is not algebraic—it is not the root of any polynomial with integer coefficients.

Hierarchy:

$$\text{Transcendental} \subset \text{Irrational} \subset \mathbb{R}. \tag{50}$$

Every transcendental number is irrational, but not every irrational number is transcendental.

Example 10.1. • $\sqrt{2}$ is algebraic (degree 2): it satisfies $x^2 - 2 = 0$.

- $\sqrt[3]{2}$ is algebraic (degree 3): it satisfies $x^3 - 2 = 0$.
- φ is algebraic (degree 2): it satisfies $x^2 - x - 1 = 0$.
- e is transcendental (Hermite, 1873).
- π is transcendental (Lindemann, 1882).

Remark 10.1. Lindemann's proof that π is transcendental also settled the ancient Greek problem of "squaring the circle"—constructing a square with the same area as a given circle using compass and straightedge. Since only algebraic numbers (of degree a power of 2) are constructible, and π is transcendental, squaring the circle is impossible.

10.2 Liouville Numbers

Definition 10.3 (Liouville Number). A real number α is a Liouville number if for every positive integer n, there exist integers p and q with $q > 1$ such that

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}. \tag{51}$$

Theorem 10.1 (Liouville, 1844). Every Liouville number is transcendental.

Example 10.2 (Liouville's Constant).

$$L = \sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001\ldots \tag{52}$$

is a Liouville number, and hence transcendental. The nonzero digits appear at positions $1!, 2!, 3!, 4!, \ldots = 1, 2, 6, 24, 120, \ldots$

10.3 Irrationality Measure

Definition 10.4 (Irrationality Measure). The irrationality measure (or irrationality exponent) of a real number α is

$$\mu(\alpha) = \inf\left\{\mu \geq 1 : \left|\alpha - \frac{p}{q}\right| > \frac{1}{q^\mu} \text{ for all but finitely many } \frac{p}{q}\right\}. \tag{53}$$

Theorem 10.2. • If α is rational, then $\mu(\alpha) = 1$.

- If α is algebraic and irrational, then $\mu(\alpha) = 2$ (Roth's theorem, 1955).
- $\mu(e) = 2$.
- $\mu(\pi) \leq 7.6063\ldots$ (Zeilberger and Zudilin, 2020).
- Liouville numbers have $\mu = \infty$.

Remark 10.2. The irrationality measure quantifies "how irrational" a number is. Rational numbers are perfectly approximable (measure 1). Most irrational numbers have measure 2. Numbers with high irrationality measure are exceptionally well-approximable by rationals.

11 Which Method Should You Use?

Number	Best Method	Key Idea
$\sqrt{n}$ (non-perfect square)	Pythagorean	Parity / prime divisibility
$\log_p q$ (p, q distinct primes)	Pythagorean	Unique factorization
$\sqrt{2} + \sqrt{3}$	Algebraic manipulation	Reduce to known irrational
e	Power series (Fourier)	Factorial tail argument
π	Niven's integral	Polynomial integral trick
Quadratic irrationals	Continued fractions	Periodicity
General algebraic irrationals	Continued fractions	Non-termination

Table 2: Choosing the right method for different numbers.

12 Why Irrationality Matters

Irrationality proofs are more than mathematical curiosities. They reveal fundamental truths about the structure of the real number line.

1. Geometry. The Pythagorean discovery of $\sqrt{2}$ showed that geometry and arithmetic are not perfectly aligned: the diagonal of a unit square has no "common measure" with the sides. This required a complete rethinking of the Greek theory of proportions.

2. Calculus. The number e is the base of the natural logarithm, the unique function equal to its own derivative, and the cornerstone of analysis. Its irrationality is essential to the structure of exponential growth and decay.

3. Physics. Both e and π appear throughout physics—in wave equations, quantum mechanics, thermodynamics, and general relativity. Their irrationality means that physical constants often cannot be expressed as simple fractions.

4. Music. The equal-tempered musical scale uses frequency ratios of $2^{k/12}$ for $k = 0, 1, \dots, 11$. These are all irrational (except $k = 0$ and $k = 12$), which means no interval in equal temperament is a "pure" ratio.

5. Number theory. Techniques from irrationality proofs—divisibility, unique factorization, Bézout's identity, continued fractions—are foundational tools used throughout modern number theory.

6. Logic and proof. Proof by contradiction, the method underlying all irrationality proofs, is one of the most powerful techniques in all of mathematics. Understanding irrationality proofs develops one's ability to construct rigorous logical arguments.

13 Conclusion

We have surveyed four major families of irrationality proofs:

1. The Pythagorean approach uses divisibility and parity arguments. It is the oldest and most elementary method, perfectly suited for square roots of non-perfect squares and logarithms.

2. Bézout's identity provides an elegant shortcut that reduces irrationality to the observation that certain quantities are not integers.

3. Power series methods exploit the specific series definitions of numbers like e, using factorial denominators to construct quantities that are simultaneously positive integers and less than 1.
4. Continued fractions offer both a theoretical characterization (rational $\iff$ finite continued fraction) and a practical tool for proving irrationality.

Additionally, Niven's integral method provides a self-contained proof of the irrationality of π using only calculus.

Each method illuminates a different aspect of the structure of irrational numbers. Together, they form a rich toolkit that connects elementary arithmetic to advanced analysis and number theory.

The study of irrationality is far from complete. Open questions remain about the irrationality of numbers such as $\pi + e$, $\pi \cdot e$, π^e, Euler's constant $\gamma \approx 0.5772\ldots$, and many values of the Riemann zeta function. These problems continue to drive research at the frontier of mathematics.

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A Summary of Key Irrationality Results

Number	Status	Method	First Proved By
$\sqrt{2}$	Irrational, algebraic	Pythagorean	Pythagoreans (c. 500 BC)
$\sqrt{n}$ (non-p.s.)	Irrational, algebraic	Pythagorean	Euclid (c. 300 BC)
φ	Irrational, algebraic	Multiple	Ancient
e	Irrational, transcendental	Series / CF	Euler (1737)
π	Irrational, transcendental	CF / Integral	Lambert (1761)
$\log_2 3$	Irrational, transcendental	Pythagorean	Folklore
e^2	Irrational, transcendental	Series	Euler (1737)
$\sqrt{2}+\sqrt{3}$	Irrational, algebraic	Algebraic	Elementary
γ	Unknown!	—	—
$\pi+e$	Unknown!	—	—

Table 3: Summary of irrationality status for notable constants. "p.s." = perfect square; "CF" = continued fractions.

B Proof Checklist: Is α Irrational?

A practical guide for deciding which method to attempt:

1. Is $\alpha = \sqrt{n}$ for some non-perfect-square integer n?
   $\longrightarrow$ Use the Pythagorean method (Theorem 4.4).
2. Is $\alpha = \log_a b$ for multiplicatively independent a, b?
   $\longrightarrow$ Use the unique factorization argument.
3. Is α a combination of known irrationals (e.g., $\sqrt{2}+\sqrt{3}$)?
   $\longrightarrow$ Algebraic manipulation to reduce to a known result.
4. Does α have a known series with factorial-type denominators?
   $\longrightarrow$ Use the Fourier/series method (Theorem 6.1).
5. Does α have a known continued fraction expansion?
   $\longrightarrow$ Show it is infinite (Theorem 7.1).
6. Is α related to π or trigonometric values?
   $\longrightarrow$ Consider Niven's integral method (Theorem 8.1).
7. None of the above?
   $\longrightarrow$ The problem may be open. Good luck!

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