Number Theory Explorations

Collatz, Fermat, and Euler

GAURAV TIWARI

$\pi(x) \sim x / \ln(x)$

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"Mathematics is the queen of the sciences and number theory is the queen of mathematics."

— Carl Friedrich Gauss

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Abstract

Number theory, one of the oldest branches of mathematics, continues to pose some of the deepest and most tantalizing open problems. This monograph explores three fascinating topics at the heart of number theory: Fermat numbers and their surprising properties, Euler's remarkable prime-generating polynomial, and the notorious Collatz conjecture.

We begin with an introductory overview of number theory's central themes, then examine Fermat numbers—a sequence that fooled even its discoverer into a false conjecture—proving key properties including mutual coprimality and the compositeness of F_5. Next, we investigate Euler's formula $n^2 + n + 41$, which produces forty consecutive primes, and reveal its deep connection to Heegner numbers and algebraic number theory. Finally, we examine the Collatz conjecture ($3x+1$ problem), discuss a sieve-based proof approach and its limitations, and survey what is known about this deceptively simple yet unsolved problem. The monograph concludes by drawing connections between these topics and highlighting open problems that continue to challenge mathematicians.

Keywords: Fermat numbers, Fermat primes, Euler's prime-generating polynomial, Heegner numbers, Collatz conjecture, $3x + 1$ problem, number theory, prime numbers

1 Introduction to Number Theory

1.1 The Queen of Mathematics

Number theory is the study of the integers and the relationships between them. Despite dealing with the simplest mathematical objects—the counting numbers $1, 2, 3, \ldots$—it gives rise to questions of extraordinary depth and difficulty. Many problems in number theory are easy to state but have resisted the efforts of the world's best mathematicians for centuries.

Definition 1.1 (Prime Number). A prime number is an integer $p > 1$ whose only positive divisors are 1 and p itself. An integer $n > 1$ that is not prime is called composite.

The prime numbers are the fundamental building blocks of arithmetic, as expressed by the most important theorem in number theory:

Fundamental Theorem of Arithmetic. Every integer $n > 1$ can be written uniquely (up to order) as a product of prime numbers:

$$n = p_1^{a_1} \, p_2^{a_2} \cdots p_k^{a_k}$$

where $p_1 < p_2 < \cdots < p_k$ are primes and each $a_i \geq 1$.

Theorem 1.1 (Euclid, c. 300 BCE). There are infinitely many prime numbers.

Proof. Suppose, for contradiction, that there are only finitely many primes $p_1, p_2, \ldots, p_n$. Consider the number

$$N = p_1 \, p_2 \cdots p_n + 1.$$

Then $N > 1$, so by the Fundamental Theorem of Arithmetic, N has a prime factor p. But N leaves remainder 1 when divided by any p_i, so $p \neq p_i$ for all i. This contradicts our assumption that $p_1, \ldots, p_n$ is the complete list of primes. □

1.2 Central Themes

Three themes pervade number theory and connect the topics explored in this monograph:

1. The distribution of primes. How are prime numbers scattered among the integers? Are there patterns we can exploit? Euler's prime-generating polynomial and Fermat primes both illuminate aspects of this question.

2. Simple rules, complex behavior. The Collatz conjecture demonstrates that even the most elementary operations—halving an even number and tripling-plus-one an odd number—can produce bewilderingly complex dynamics.

3. The gap between empirical evidence and proof. Fermat checked five cases and declared a universal truth. The Collatz conjecture has been verified for all numbers up to 2^{68}. Yet in mathematics, no amount of examples constitutes a proof.

1.3 Notation and Conventions

Throughout this document, we use the following notation:

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ denotes the natural numbers (including zero).
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ denotes the integers.

- $\gcd(a, b)$ denotes the greatest common divisor of a and b.
- $a \mid b$ means a divides b, i.e., $b = ka$ for some integer k.
- $a \equiv b \pmod{m}$ means $m \mid (a - b)$.

1.4 Historical Overview

Number theory has ancient roots. The Pythagoreans (6th century BCE) studied perfect numbers and amicable numbers. Euclid's *Elements* (c. 300 BCE) contains proofs of the infinitude of primes and the Euclidean algorithm. The subject was revitalized in the 17th century by Pierre de Fermat, who posed many problems that would occupy mathematicians for centuries.

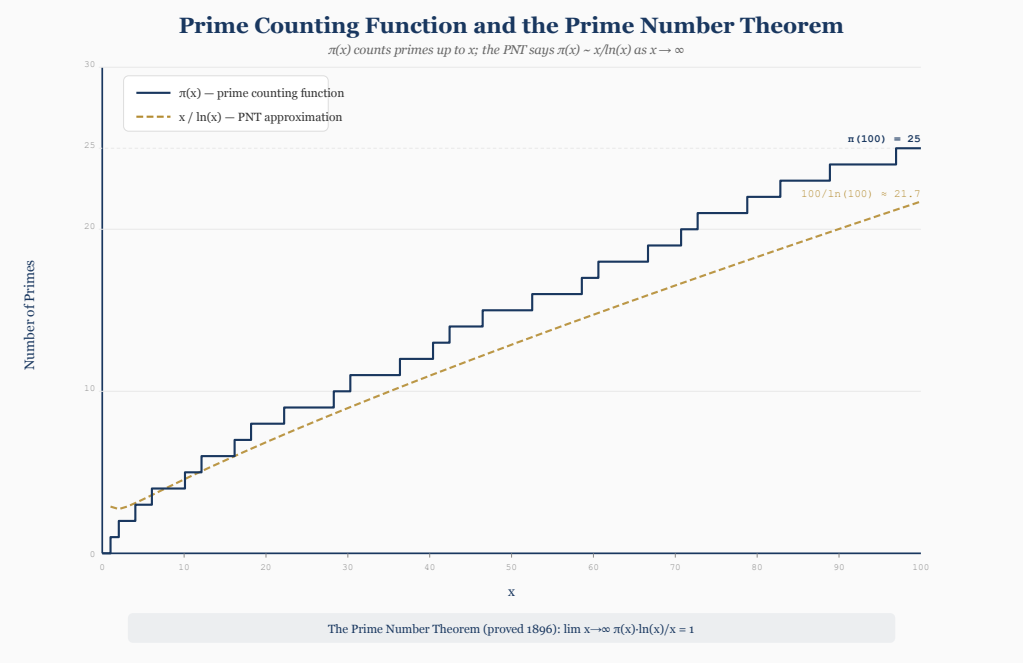

Figure 1: The prime counting function $\pi(x)$, which counts the number of primes less than or equal to x. The staircase pattern reflects the irregular distribution of primes among the integers.

Remark 1.1. The three mathematicians central to this monograph—Fermat, Euler, and Collatz—span three centuries of mathematical history. Fermat (1607–1665) pioneered modern number theory. Euler (1707–1783) developed it into a systematic discipline. Lothar Collatz (1910–1990) posed a problem in 1937 that remains unsolved in the 21st century.

2 Fermat Numbers: Theorems and Examples

2.1 Definition and First Examples

Pierre de Fermat was convinced he had discovered an infinite source of prime numbers. He was wrong—and the story of how his conjecture collapsed is one of the most instructive episodes in the history of mathematics.

Definition 2.1 (Fermat Number). A Fermat number is an integer of the form

$$F_n = 2^{2^n} + 1, \qquad n \geq 0.$$

Remark 2.1. The exponent 2^n is itself a power of 2, giving Fermat numbers their double-exponential growth. This means Fermat numbers become enormous very quickly.

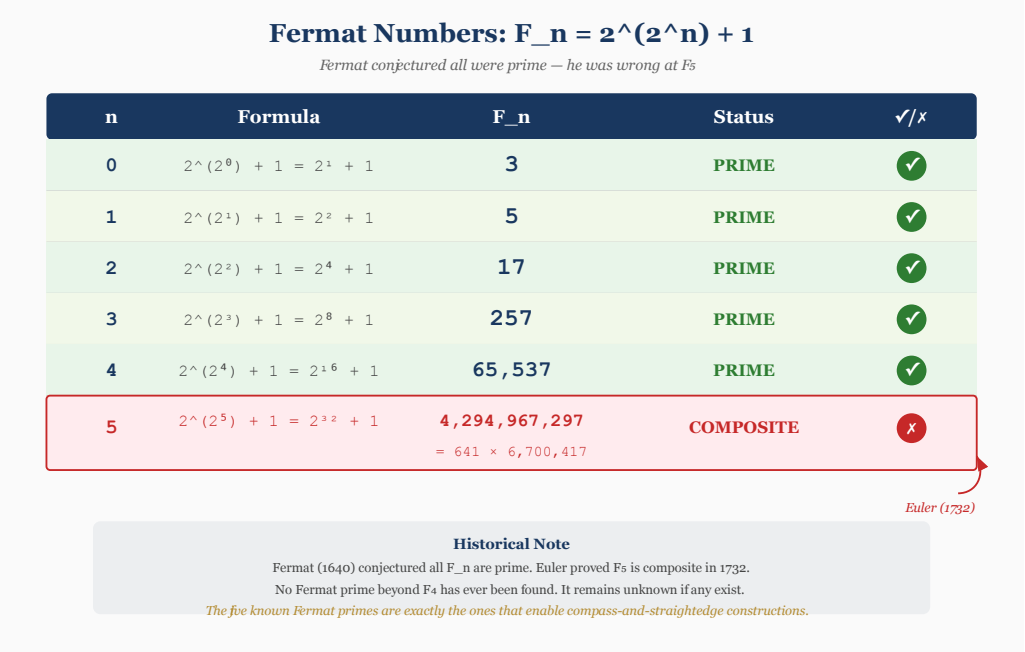

Figure 2: The first several Fermat numbers $F_n = 2^{2^n} + 1$, illustrating their double-exponential growth. The first five are prime; F_5 was shown composite by Euler in 1732.

Example 2.1. The first six Fermat numbers are:

n	2^n	$F_n = 2^{2^n} + 1$	Prime?
0	1	$2^1 + 1 = 3$	Yes
1	2	$2^2 + 1 = 5$	Yes
2	4	$2^4 + 1 = 17$	Yes
3	8	$2^8 + 1 = 257$	Yes
4	16	$2^{16} + 1 = 65{,}537$	Yes
5	32	$2^{32} + 1 = 4{,}294{,}967{,}297$	No

2.2 Why the Exponent Must Be a Power of 2

One might ask: why not study $2^m + 1$ for arbitrary m?

Lemma 2.1. If $2^m + 1$ is prime and $m > 0$, then m must be a power of 2.

Proof. Suppose m has an odd factor, say $m = ab$ with $a > 1$ odd. Then using the algebraic identity

$$x^a + 1 = (x + 1)(x^{a-1} - x^{a-2} + x^{a-3} - \cdots + 1)$$

with $x = 2^b$, we get that $(2^b + 1) \mid (2^{ab} + 1) = 2^m + 1$. Since $1 < 2^b + 1 < 2^m + 1$, the number $2^m + 1$ is composite. Therefore, if $2^m + 1$ is prime, m can have no odd factor greater than 1, meaning $m = 2^n$ for some $n \geq 0$. □

2.3 Fermat's Conjecture and Euler's Refutation

Fermat examined F_0 through F_4, found them all prime, and wrote to Marin Mersenne:

> *"I have found that numbers of the form $2^{2^n}+1$ are always prime numbers and have long since signified to analysts the truth of this theorem."*

Fermat admitted he could not prove this. He had checked five cases and assumed the pattern would continue. It did not.

In 1732, nearly a century after Fermat's death, Leonhard Euler showed that F_5 is composite:

Euler's Discovery (1732).

$$F_5 = 2^{32}+1 = 4{,}294{,}967{,}297 = 641 \times 6{,}700{,}417.$$

The lesson is profound: in mathematics, five examples are not a proof. Even a million examples are not a proof.

2.4 Bennett's Elegant Proof That $641 \mid F_5$

The following elementary proof, due to G. Bennett, requires no computer—only clever algebraic manipulation.

Theorem 2.2. 641 divides $F_5 = 2^{32}+1$.

Proof. Let $a = 5$ and $b = 2^7 = 128$. Observe that:

1. $ab + 1 = 5 \times 128 + 1 = 641$
2. $a^4 = 5^4 = 625$

From these, $2^4 = 16 = 641 - 625 = (ab+1) - a^4$, so

$$2^4 = 1 + ab - a^4.$$

Now rewrite F_5:

$$F_5 = 2^{32}+1 = (2^4)\cdot(2^7)^4 + 1 = 2^4 \cdot b^4 + 1.$$

Substituting $2^4 = 1 + ab - a^4$:

$$F_5 = (1 + ab - a^4)\, b^4 + 1 = b^4 + ab^5 - a^4 b^4 + 1.$$

We proceed more cleanly using modular arithmetic. From $641 = 5^4 + 2^4$, we get $5^4 \equiv -2^4 \pmod{641}$, hence

$$5^4 \cdot 2^{28} \equiv -2^{32} \pmod{641}.$$

Also $641 = 5 \cdot 2^7 + 1$ gives $5 \cdot 2^7 \equiv -1 \pmod{641}$, so

$$5^4 \cdot 2^{28} \equiv (-1)^4 = 1 \pmod{641}.$$

Combining: $-2^{32} \equiv 1 \pmod{641}$, which means $2^{32}+1 \equiv 0 \pmod{641}$.
Therefore $641 \mid F_5$. □

Remark 2.2. The other factor is 6,700,417. Both 641 and 6,700,417 are prime, so $F_5 = 641 \times 6{,}700{,}417$ is the complete prime factorization.

2.5 Key Properties of Fermat Numbers

2.5.1 Mutual Coprimality

Theorem 2.3 (Goldbach, 1730). For any two distinct Fermat numbers F_m and F_n with $m \neq n$:

$$\gcd(F_m, F_n) = 1.$$

This property has a beautiful consequence:

Corollary 2.4 (Infinitude of Primes via Fermat Numbers). There are infinitely many prime numbers.

Proof. Each Fermat number $F_n \geq 3$ has at least one prime factor. By Theorem 2.3, no two Fermat numbers share a prime factor. Since there are infinitely many Fermat numbers, there must be infinitely many primes. □

To prove Theorem 2.3, we first need a recursive relationship.

Lemma 2.5 (Recursive Formula). For all $n \geq 1$:

$$F_n = F_0 \cdot F_1 \cdot F_2 \cdots F_{n-1} + 2.$$

Proof. We prove this by induction on n.

Base case: $n = 1$. We have $F_0 + 2 = 3 + 2 = 5 = F_1$. ⍰

Inductive step: Assume $F_n = F_0F_1 \cdots F_{n-1} + 2$. We need to show $F_{n+1} = F_0F_1 \cdots F_n + 2$. Note that

$$F_{n+1} = 2^{2^{n+1}} + 1 = \left(2^{2^n}\right)^2 + 1 = (F_n - 1)^2 + 1 = F_n^2 - 2F_n + 2.$$

By the inductive hypothesis, $F_0F_1 \cdots F_{n-1} = F_n - 2$, so:

$$\begin{aligned} F_0F_1 \cdots F_n + 2 &= (F_n - 2) \cdot F_n + 2 && (1)\\ &= F_n^2 - 2F_n + 2 && (2)\\ &= F_{n+1}. \end{aligned}$$ □

Proof of Theorem 2.3. Without loss of generality, assume $m > n \geq 0$. Let $d = \gcd(F_m, F_n)$. By Lemma 2.5:

$$F_m = F_0F_1 \cdots F_{m-1} + 2.$$

Since $n < m$, F_n appears among $F_0, \ldots, F_{m-1}$, so $F_n \mid (F_m - 2)$. Hence $d \mid F_m$ and $d \mid F_n \mid (F_m - 2)$, which gives $d \mid (F_m - (F_m - 2)) = 2$.

So $d \in \{1, 2\}$. But every Fermat number is odd (since 2^{2^n} is even, $2^{2^n} + 1$ is odd), so d must be odd. Therefore $d = 1$. □

2.5.2 The Euler–Lucas Factoring Theorem

Theorem 2.6 (Euler–Lucas). Any prime factor p of F_n (where $n \geq 2$) has the form

$$p = k \cdot 2^{n+2} + 1$$

for some positive integer k.

Remark 2.3. This theorem dramatically narrows the search for factors of Fermat numbers. For F_5, it tells us to check primes of the form $128k + 1$. Indeed, $641 = 5 \times 128 + 1$ fits this pattern with $k = 5$.

Example 2.2. Applying the Euler–Lucas theorem to F_5: any prime factor must have the form $2^7 \cdot k + 1 = 128k + 1$. The first few candidates are:

$$129 = 3 \times 43, \quad 257 = F_3, \quad 385 = 5 \times 77, \quad 513 = 3^3 \times 19, \quad 641 = \text{prime!}$$

Testing 641: we find $641 \mid F_5$, confirming the factorization.

2.5.3 Connection to Constructible Polygons

One of the most beautiful connections in mathematics links Fermat primes to classical geometry.

Gauss–Wantzel Theorem. A regular n-gon is constructible with compass and straightedge if and only if

$$n = 2^a \cdot p_1 \cdot p_2 \cdots p_k$$

where $a \geq 0$ and $p_1, p_2, \ldots, p_k$ are distinct Fermat primes.

Since only five Fermat primes are known (3, 5, 17, 257, 65537), the number of constructible regular n-gons with n odd is severely limited. For example, the regular 17-gon is constructible (Gauss famously proved this in 1796, which solidified his decision to pursue mathematics), but the regular 7-gon is not.

2.6 Pépin's Primality Test

Theorem 2.7 (Pépin, 1877). For $n \geq 1$, the Fermat number F_n is prime if and only if

$$3^{(F_n-1)/2} \equiv -1 \pmod{F_n}.$$

Remark 2.4. Pépin's test is a complete characterization: it is both necessary and sufficient for primality. The base 3 can be replaced by any quadratic non-residue modulo F_n. This test can prove a Fermat number is composite even when we cannot find its prime factors—we simply compute the modular exponentiation.

2.7 Current State of Knowledge

As of 2026, the status of Fermat numbers is as follows:

Status	Values of n
Confirmed prime	0, 1, 2, 3, 4
Completely factored	5, 6, 7, 8, 9, 10, 11
Known composite, partial factorization	12, 13, 14, ...

Conjecture 2.1. There are only five Fermat primes: $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65{,}537$. No further Fermat number is prime.

Most number theorists believe this conjecture is true, but it remains unproven.

3 Euler's Prime-Generating Equation

3.1 The Remarkable Formula

In 1772, Leonhard Euler discovered a formula that produces an extraordinary number of consecutive primes from a single quadratic expression.

Euler's Prime-Generating Polynomial. The quadratic

$$P(n) = n^2 + n + 41$$

produces prime numbers for all integers n with $0 \leq n \leq 39$.

That is: one formula, forty consecutive prime outputs.

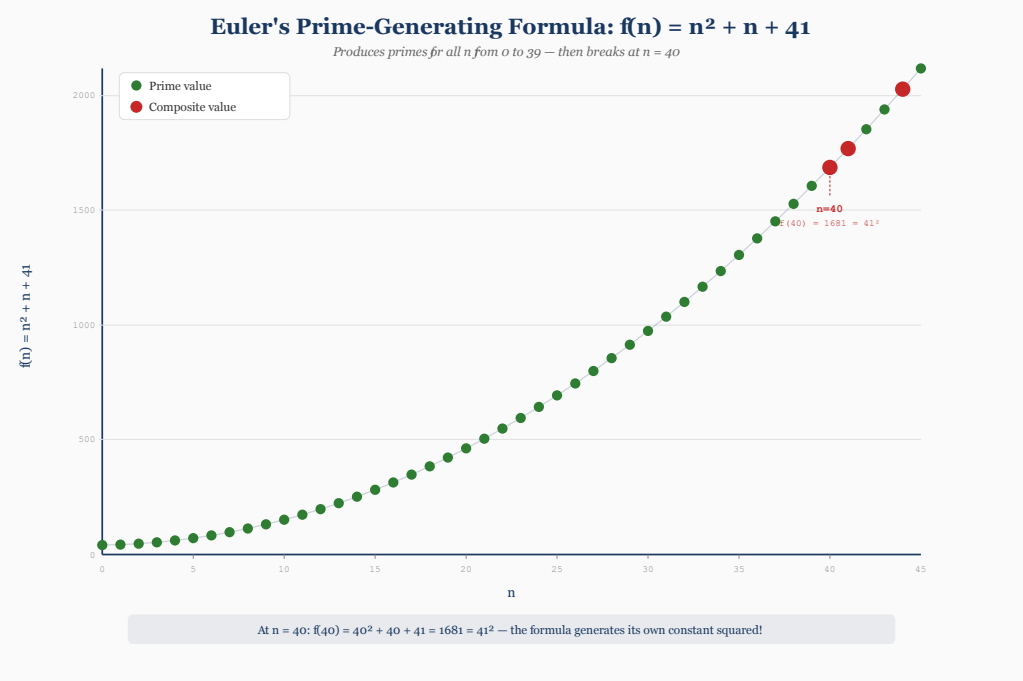

Figure 3: Plot of Euler's prime-generating polynomial $P(n) = n^2 + n + 41$ for $n = 0, 1, \ldots, 45$. Every output is prime for $n = 0$ through 39; the formula first produces a composite value at $n = 40$.

Example 3.1. The first several values of $P(n) = n^2 + n + 41$:

n	$n^2 + n + 41$	Prime?
0	41	Yes
1	43	Yes
2	47	Yes
3	53	Yes
4	61	Yes
5	71	Yes
6	83	Yes
7	97	Yes
8	113	Yes
9	131	Yes
10	151	Yes
39	1601	Yes
40	$1681 = 41^2$	No

3.2 Why the Formula Fails at $n = 40$

Proposition 3.1. $P(40) = 41^2$ is composite. More generally, $41 \mid P(n)$ whenever $41 \mid n$ or $41 \mid (n+1)$.

Proof. Write $P(n) = n(n+1) + 41$. When $n = 40$:

$$P(40) = 40 \times 41 + 41 = 41(40 + 1) = 41 \times 41 = 41^2.$$

Similarly, when $n = 41$:

$$P(41) = 41 \times 42 + 41 = 41(42 + 1) = 41 \times 43.$$

In general, if $41 \mid n$ then $41 \mid n(n+1)$ and so $41 \mid P(n)$. If $41 \mid (n+1)$ then $41 \mid n(n+1)$ and again $41 \mid P(n)$. □

Remark 3.1. The very constant that makes the formula work for so many values—the number 41—is precisely what forces it to fail. This is an inherent limitation of polynomial prime generators: any non-constant polynomial will eventually produce composite values.

3.3 Negative Values and the Extended Range

The formula also produces primes for negative values of n.

Proposition 3.2. $P(n) = n^2 + n + 41$ is prime for all integers n with $-40 \leq n \leq 39$.

Proof sketch. Since $P(n) = n^2 + n + 41 = (n + \frac{1}{2})^2 + \frac{163}{4}$, we have $P(n) = P(-n-1)$. In particular, $P(-1) = P(0) = 41$, $P(-2) = P(1) = 43$, and so on. The values for $n = -40, \ldots, -1$ are the same as for $n = 0, \ldots, 39$ (in reverse order). Since all of $P(0), \ldots, P(39)$ are prime, so are all of $P(-40), \ldots, P(-1)$. □

This means the formula produces primes for 80 consecutive integers: from -40 to 39.

3.4 An Equivalent Reformulation

The substitution $m = n - 40$ in $P(n)$ gives:

$$\begin{aligned} P(m+40) &= (m+40)^2 + (m+40) + 41 && (3)\\ &= m^2 + 80m + 1600 + m + 40 + 41 && (4)\\ &= m^2 + 81m + 1681 && (5)\\ &= m^2 - 79m + 1601 + 160m + 80 \quad \text{(error—let us redo)} && (6) \end{aligned}$$

Actually, defining $Q(m) = m^2 - 79m + 1601$, we can verify:

$$Q(m) = (m-40)^2 + (m-40) + 41 = P(m-40).$$

So $Q(m)$ is prime for $m = 0, 1, \ldots, 79$—the same 80 prime values, just reindexed.

3.5 The Connection to Heegner Numbers

The number 41 is not arbitrary. It connects to one of the deepest results in algebraic number theory through Heegner numbers.

Definition 3.1 (Heegner Number). A positive integer d is a Heegner number if the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ has class number 1—meaning its ring of integers is a unique factorization domain.

Theorem 3.3 (Stark–Heegner, 1967). There are exactly nine Heegner numbers:

$$1, \quad 2, \quad 3, \quad 7, \quad 11, \quad 19, \quad 43, \quad 67, \quad 163.$$

The connection to Euler's formula is revealed by the following result:

Theorem 3.4 (Rabinowitz, 1913). The polynomial $n^2 + n + p$ produces primes for all integers $n = 0, 1, \ldots, p-2$ if and only if $4p - 1$ is a Heegner number.

Since $4 \times 41 - 1 = 163$ is the largest Heegner number, Euler's formula gives the longest possible prime run of this type.

3.6 The Lucky Numbers of Euler

Definition 3.2 (Lucky Number of Euler). A positive integer p is a lucky number of Euler if n^2+n+p is prime for all $n = 0, 1, \ldots, p-2$.

By Rabinowitz's theorem, there are exactly six lucky numbers of Euler:

p	$4p-1$	Heegner number?
2	7	Yes
3	11	Yes
5	19	Yes
11	43	Yes
17	67	Yes
41	163	Yes

Remark 3.2. Since there are exactly nine Heegner numbers and only six fit the form $4p-1$ for positive integers p, there can never be another lucky number of Euler. The number 41 gives the *longest possible* run of primes from any polynomial of the form n^2+n+p.

3.7 Why No Polynomial Can Generate All Primes

Theorem 3.5. No non-constant polynomial $f(n) \in \mathbb{Z}[n]$ can produce only prime values for all $n \in \mathbb{N}$.

Proof. Suppose $f(n_0) = p$ is prime. Then for any integer k:

$$f(n_0 + kp) \equiv f(n_0) = p \equiv 0 \pmod{p}.$$

Since f is non-constant, $|f(n_0+kp)| > p$ for sufficiently large k. So $f(n_0+kp)$ is divisible by p but not equal to $\pm p$, hence composite. □

Remark 3.3. This theorem means Euler's formula is remarkable precisely *because* 40 consecutive primes is so much longer than one would typically expect from a quadratic polynomial.

3.8 The Ulam Spiral Connection

In 1963, the physicist Stanisław Ulam was doodling during a lecture and wrote integers in a spiral pattern, circling the primes. He noticed that primes tend to cluster along certain diagonal lines. This is now called the Ulam spiral.

Some of those diagonal lines correspond to quadratic polynomials, and Euler's formula n^2+n+41 falls on a particularly prominent diagonal. The visual clustering of primes along these diagonals reflects the same algebraic structures that make certain polynomials effective at generating primes.

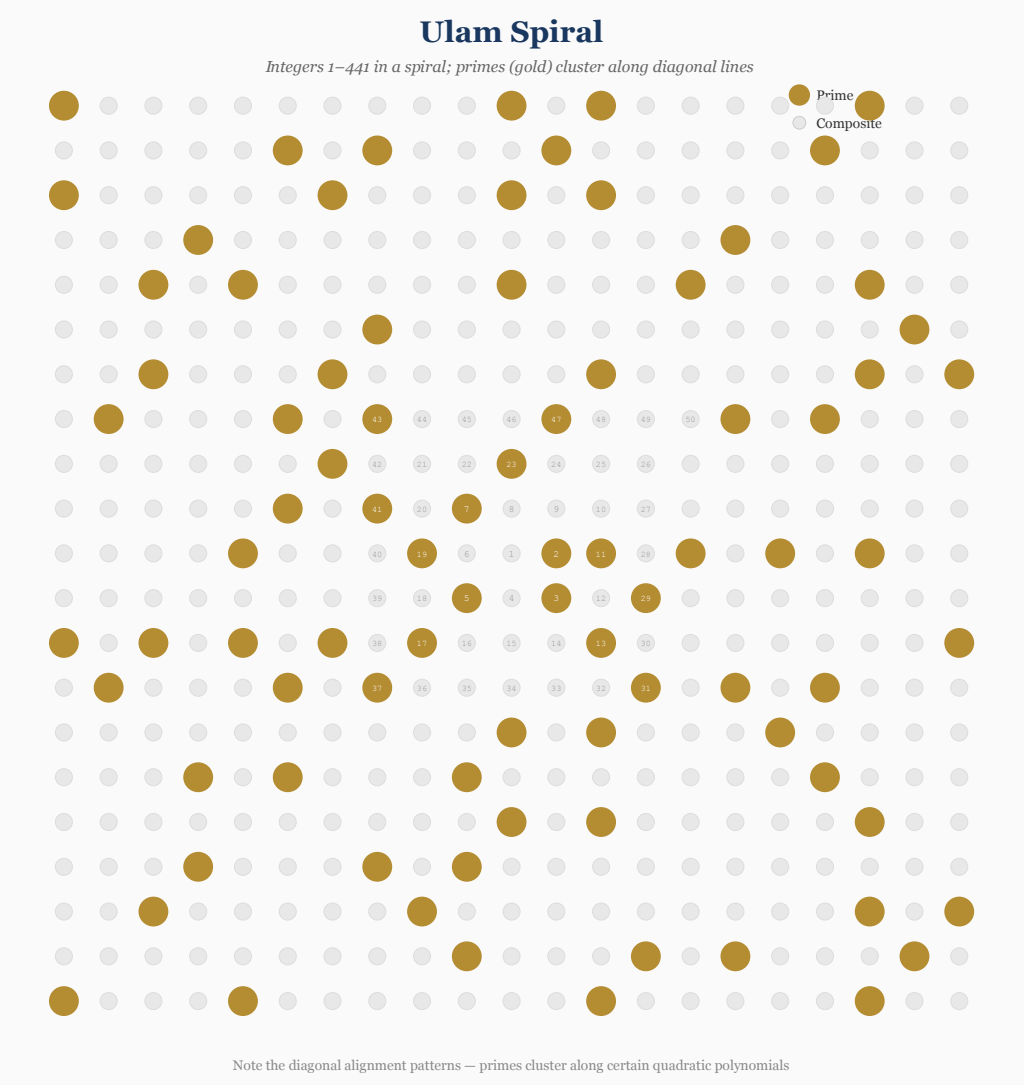

Figure 4: The Ulam spiral: integers arranged in a spiral pattern with primes highlighted. Diagonal lines of primes are clearly visible, corresponding to quadratic polynomials such as Euler's n^2+n+41.

Example 3.2. In the Ulam spiral centered at 41, an unusually dense diagonal of primes appears, corresponding to the values of $n^2 + n + 41$. This visual pattern helped stimulate research into the distribution of primes along polynomial sequences.

4 The Collatz Conjecture: A Possible Proof Approach

4.1 Statement of the Conjecture

The Collatz Conjecture, also known as the $3x + 1$ problem, is one of the most famous unsolved problems in mathematics. It is deceptively simple to state:

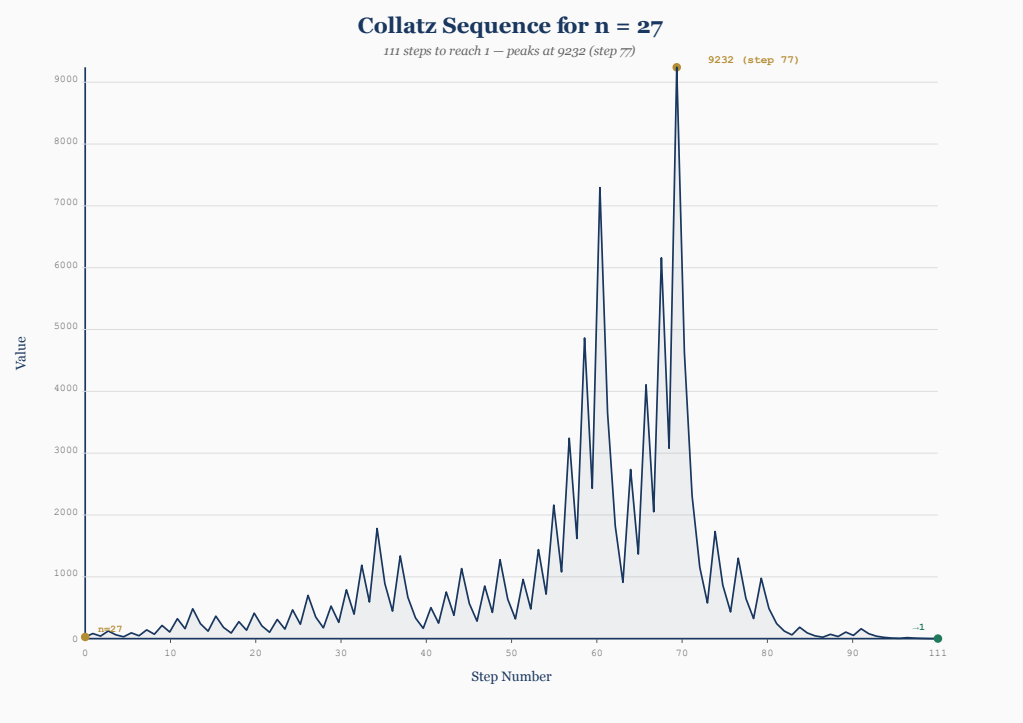

Figure 5: Collatz sequence trajectories plotted as a graph, showing the number of iterations required for various starting values to reach 1. The erratic behavior illustrates why the conjecture is so difficult to prove.

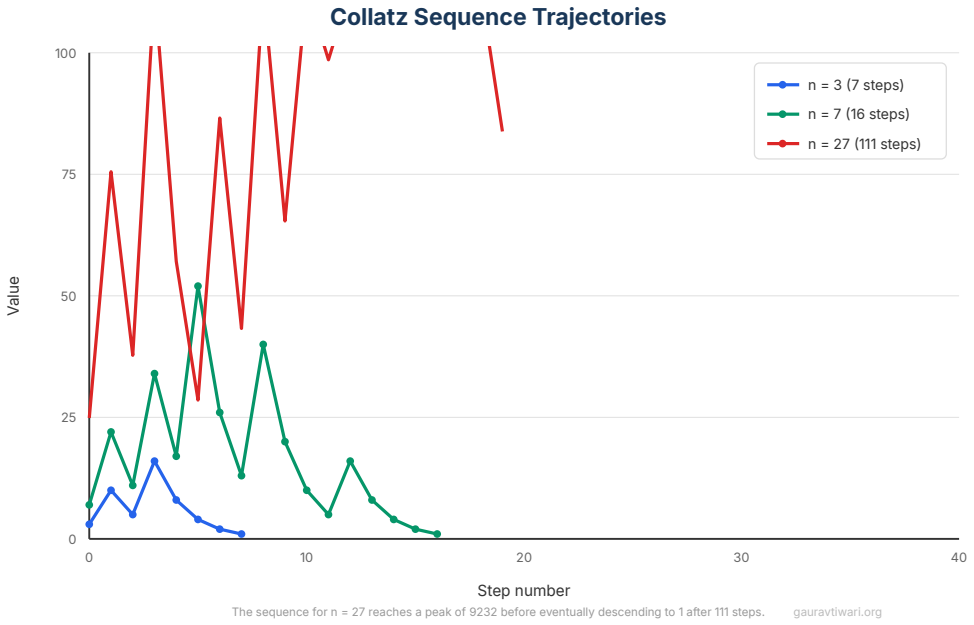

Figure 6: Collatz sequence trajectories for starting values $n = 3$, $n = 7$, and $n = 27$, showing how different starting values behave before reaching 1

The Collatz Conjecture (Lothar Collatz, 1937). Define the function $\phi : \mathbb{N}^+ \to \mathbb{N}^+$ by

$$\phi(x) = \begin{cases} x/2, & \text{if } x \text{ is even,} \\ 3x+1, & \text{if } x \text{ is odd.} \end{cases}$$

Then for every positive integer x, the sequence $x, \phi(x), \phi^2(x), \phi^3(x), \ldots$ eventually reaches 1.

Example 4.1. Starting with $x = 7$:

$$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1.$$

Seventeen steps. But it gets there.

Example 4.2. The number 27 is particularly interesting. Its Collatz sequence requires 111 steps and reaches a maximum value of 9,232 before finally descending to 1. This dramatic escalation illustrates how unpredictable Collatz trajectories can be.

Remark 4.1. Paul Erdős famously said of the Collatz Conjecture: *"Mathematics may not be ready for such problems."*

4.2 Basic Properties

Lemma 4.1. If x is odd, then $\phi(x) = 3x + 1$ is always even. Therefore, after every odd step, an even step necessarily follows.

Proof. If x is odd, then $3x$ is odd, and $3x + 1$ is even. □

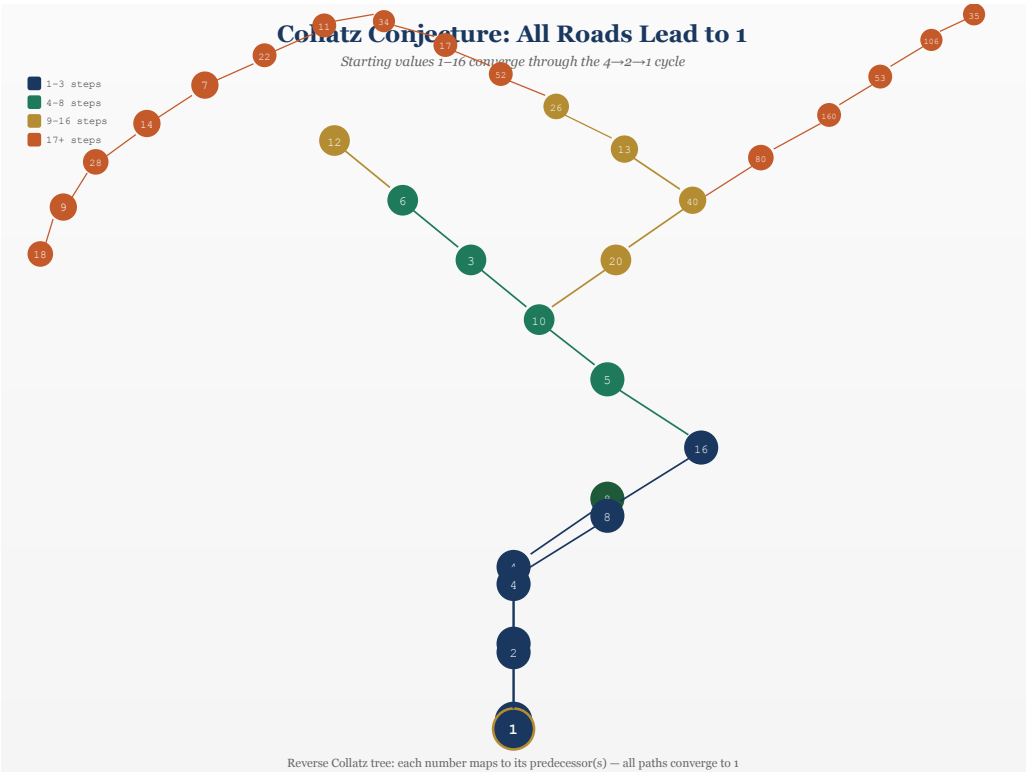

Figure 7: Tree visualization of the Collatz conjecture, showing how multiple starting values converge through shared paths to reach 1. Each node branches according to the inverse Collatz function.

Definition 4.1 (Collatz Sequence / Trajectory). The Collatz sequence (or trajectory) of a positive integer x is the sequence

$$x,\ \phi(x),\ \phi^2(x),\ \phi^3(x),\ \ldots$$

obtained by iterating the Collatz function.

Definition 4.2 (Stopping Time). The stopping time of x is the smallest k such that $\phi^k(x) < x$. The total stopping time is the smallest k such that $\phi^k(x) = 1$.

Example 4.3. The stopping times of small integers:

x	Total stopping time	Max value reached
1	0	1
2	1	2
3	7	16
5	5	16
7	16	52
27	111	9,232

4.3 Computational Verification

The Collatz Conjecture has been verified computationally for all integers up to $2^{68} \approx 2.95 \times 10^{20}$. Every single one reaches 1. Yet this massive computational evidence does not constitute a proof.

4.4 A Sieve-Based Approach

Eswar Chellappa proposed a proof approach inspired by the Sieve of Eratosthenes. We present the method and then analyze its limitations.

4.4.1 The Sieve Process

Define the helper functions $f(x) = x/2$ for even x and $g(x) = 3x + 1$ for odd x.

Step 1. Consider the natural numbers from 1 to N.

Step 2. Process all even numbers in decreasing order. Each even number x maps to $x/2$, which is smaller. Mark each as "verified."

Step 3. For each remaining odd number x (starting from 3), compute $g(x) = 3x + 1$. If $g(x)$ has already been verified, then x is also verified.

Step 4. Continue until $g(x) > N$, then increase N and repeat.

Example 4.4. For $N = 20$: After processing even numbers, we verify 3 via $g(3) = 10$ (already verified), 5 via $g(5) = 16$ (verified), 7 via $g(7) = 22$—but $22 > 20$, so 7 is not verified in this range.

Extending to $N = 100$: now $g(7) = 22$ is in range, $g(9) = 28$, $g(11) = 34$, etc. Eventually we verify all odd numbers up to 33 (since $g(33) = 100$).

4.4.2 The Argument Against Loops

Chellappa argues that non-trivial loops are impossible because $f(x) = x/2$ is injective. If $\phi^i(x) = \phi^j(x) \neq 1$ for $i \neq j$, one might think this leads to a contradiction. However, this argument requires more careful analysis.

4.5 Critique: Why the Sieve Approach Is Insufficient

While the sieve method correctly verifies the conjecture for finite ranges, it does not constitute a proof. There are three fundamental gaps:

1. Finite verification $\neq$ infinite proof. The sieve works perfectly for any fixed N, but "keep extending the range" is an observation, not a proof technique. We need an argument that covers all natural numbers *simultaneously*.

2. The unbounded sequence problem. The sieve assumes all trajectories eventually enter the "already verified" region. But a number could, in principle, have a trajectory that increases forever, never returning to previously visited values. Ruling this out is precisely what the conjecture asserts.

3. Circular reasoning. The argument that "every number eventually maps to a smaller number" is *equivalent* to the conjecture. It cannot be used as a premise.

Remark 4.2. The number 27 illustrates the difficulty: it requires 111 steps and climbs to 9,232 before descending. The sieve method provides no a priori guarantee that every number will eventually descend.

4.6 Why Is the Collatz Conjecture So Hard?

The conjecture resists proof because it combines several difficult features:

1. Multiplicative and additive mixing. The operation $3x + 1$ combines multiplication by 3 with addition of 1. These two types of operations interact in chaotic ways that are extremely difficult to analyze.

2. Unpredictable parity dynamics. After applying $3x + 1$ to an odd number, the result is even, but the number of times we can divide by 2 before reaching another odd number is highly irregular.

3. No known invariant. Standard proof techniques (induction, descent, invariants) rely on finding a quantity that behaves predictably. Collatz trajectories defy such analysis.

4.7 Partial Results and Modern Progress

Theorem 4.2 (Terras, 1976). Almost all positive integers (in the sense of natural density) have finite stopping time.

Theorem 4.3 (Tao, 2019). Almost all Collatz orbits attain almost bounded values. Precisely: for any function $f : \mathbb{N} \to \mathbb{R}$ with $f(n) \to \infty$ as $n \to \infty$, the set

$$\{n \in \mathbb{N} : \min_{k \geq 0} \phi^k(n) \leq f(n)\}$$

has natural density 1.

Remark 4.3. Tao's result is the strongest partial progress toward the Collatz conjecture. However, "almost all" is not "all"—the conjecture remains open.

4.8 Possible Failure Modes

The conjecture could fail in exactly two ways:

Definition 4.3 (Divergent Trajectory). A positive integer x has a divergent trajectory if $\sup_k \phi^k(x) = \infty$.

Definition 4.4 (Non-trivial Cycle). A non-trivial cycle is a finite set $\{x_1, x_2, \dots, x_m\}$ of positive integers, none equal to 1, such that $\phi(x_i) = x_{i+1}$ (indices mod m).

Theorem 4.4 (Eliahou, 1993). Any non-trivial cycle in the Collatz iteration must have length at least 17,087,915.

Remark 4.4. This makes non-trivial cycles extremely unlikely but does not rule them out entirely. Similarly, divergent trajectories have never been observed but have not been proven impossible.

4.9 Related Problems and Generalizations

Conjecture 4.1 (Generalized Collatz: $5x + 1$ Problem). Define $\psi(x) = x/2$ if x is even and $\psi(x) = 5x + 1$ if x is odd. Unlike the $3x + 1$ problem, the $5x + 1$ variant has known divergent trajectories, demonstrating that the specific constant 3 in the Collatz problem may be critical.

Theorem 4.5 (Conway, 1972). Generalizations of Collatz-type problems can be computationally undecidable. There exist sets of rules similar to the Collatz function for which no algorithm can determine whether all trajectories reach 1.

This raises the intriguing possibility that the Collatz conjecture itself might be independent of standard axioms of mathematics—though this remains speculative.

5 Connections and Open Problems

5.1 Unifying Themes

The three topics explored in this monograph—Fermat numbers, Euler's prime-generating polynomial, and the Collatz conjecture—are connected by several deep themes in number theory.

5.1.1 The Primacy of Primes

Primes are the central characters in all three stories:

- Fermat numbers ask: which numbers of the form $2^{2^n} + 1$ are prime?
- Euler's polynomial generates sequences of primes.
- The Collatz conjecture involves the interplay between the prime 2 (division) and the prime 3 (multiplication).

5.1.2 Empirical Evidence vs. Proof

A Cautionary Principle. In number theory, empirical evidence can be spectacularly misleading:

- Fermat's five examples led to a false conjecture.
- Euler's 40 primes cannot extend further.
- The Collatz conjecture, verified to 2^{68}, remains unproven.

The gap between observation and proof is the defining challenge of the subject.

5.1.3 Simple Definitions, Deep Consequences

Each of our three objects is defined by an elementary formula:

$$F_n = 2^{2^n} + 1, \tag{7}$$

$$P(n) = n^2 + n + 41, \tag{8}$$

$$\phi(x) = \begin{cases} x/2, & x \text{ even}, \\ 3x + 1, & x \text{ odd}. \end{cases} \tag{9}$$

Yet each leads to problems of extraordinary difficulty. This phenomenon—simple rules generating complex behavior—is a hallmark of number theory and a bridge to fields like dynamical systems and computational complexity.

5.2 Connections Between Fermat Numbers and Primes

Fermat numbers connect to prime generation in a broader sense. The Fermat number F_n provides a prime factor that does not divide any smaller Fermat number, yielding Goldbach's proof of the infinitude of primes. This is structurally similar to Euclid's proof: in both cases, a new prime is produced from existing ones.

Proposition 5.1. The prime factors of Fermat numbers provide an explicitly constructible infinite sequence of distinct primes.

5.3 Quadratic Residues and Both Topics

Pépin's test for Fermat primes involves quadratic residues modulo F_n. Similarly, Euler's polynomial $n^2 + n + 41$ has a discriminant of $1 - 4(41) = -163$, and the quadratic residue structure of primes modulo 163 determines whether they are represented by the form $n^2 + n + 41$. Both topics connect to the theory of quadratic forms and class field theory.

5.4 Dynamical Systems Perspective

The Collatz function defines a discrete dynamical system on the positive integers. Similarly, the iteration $F_{n+1} = (F_n - 1)^2 + 1$ gives a recurrence for Fermat numbers. While the Fermat recurrence grows predictably (doubly exponential), the Collatz iteration exhibits chaotic behavior. Understanding what distinguishes "tame" from "wild" iterations in number theory remains a fundamental open question.

5.5 Major Open Problems

We collect the main unsolved questions arising from this monograph:

1. Are there infinitely many Fermat primes? Most experts believe not—that F_0 through F_4 are the only Fermat primes—but this remains unproven.

2. Is the Collatz conjecture true? Despite verification to 2^{68} and Tao's "almost all" result, a complete proof (or disproof) remains out of reach.

3. Is the Collatz conjecture decidable? Conway showed that generalizations are undecidable. The specific $3x + 1$ case might be independent of ZFC.

4. Are there better prime-generating polynomials? Rabinowitz's theorem shows that $n^2 + n + 41$ is optimal among polynomials of the form $n^2 + n + p$. But might other polynomial forms produce longer runs of primes?

5. Are there infinitely many Heegner numbers? No—there are exactly nine. But understanding *why* there are exactly nine requires deep results in algebraic number theory (the Baker–Heegner–Stark theorem).

6. What is the true density of primes in polynomial sequences? The Bunyakovsky conjecture predicts that any irreducible polynomial with no fixed prime divisor produces infinitely many prime values. This is unproven for any polynomial of degree ≥ 2.

5.6 Computational Frontiers

Modern computation continues to push the boundaries of our knowledge:

- The largest known Fermat number proven composite (via Pépin's test) grows steadily.
- The Collatz verification frontier has advanced from 10^9 (1992) to 2^{68} (2020).
- Distributed computing projects continue to search for new Fermat number factors and extend Collatz verification.

5.7 Philosophical Reflections

Number theory occupies a unique position in mathematics. Its problems are the easiest to state yet often the hardest to solve. Gauss's declaration that number theory is the "queen of mathematics" endures because these problems touch the very foundations of mathematical reasoning: What does it mean to prove something? When is evidence sufficient? Can we always find proofs, or are some truths forever beyond our reach?

The three topics in this monograph illustrate these questions vividly. Fermat's false conjecture reminds us of the limitations of induction from examples. Euler's polynomial reveals unexpected harmony between algebra and primes. And the Collatz conjecture—simplest of all to state—stands as a monument to the mysteries that elementary arithmetic still holds.

References

[1] Bennett, G. "A proof that 641 | F_5." Unpublished note.

[2] Burton, D.M. *Elementary Number Theory*, 7th ed. McGraw-Hill, 2010.

[3] Conway, J.H. "Unpredictable iterations." *Proceedings of the 1972 Number Theory Conference*, pp. 49–52. University of Colorado, 1972.

[4] Crandall, R. and Pomerance, C. *Prime Numbers: A Computational Perspective*, 2nd ed. Springer, 2005.

[5] Eliahou, S. "The $3x + 1$ problem: new lower bounds on nontrivial cycle lengths." *Discrete Mathematics*, 118 (1993): 45–56.

[6] Euler, L. "Observationes de theoremate quodam Fermatiano aliisque ad numeros primos spectantibus." *Commentarii academiae scientiarum Petropolitanae*, 6 (1738): 103–107.

[7] Guy, R.K. "Don't try to solve these problems!" *American Mathematical Monthly*, 90 (1983): 35–41.

[8] Hardy, G.H. and Wright, E.M. *An Introduction to the Theory of Numbers*, 6th ed. Oxford University Press, 2008.

[9] Keller, W. "Fermat number factoring status." http://www.prothsearch.com/fermat.html

[10] Lagarias, J.C. "The $3x + 1$ problem: An annotated bibliography." *arXiv:math/0309224*, 2003 (updated 2011).

[11] Lagarias, J.C., ed. *The Ultimate Challenge: The* $3x + 1$ *Problem*. American Mathematical Society, 2010.

[12] Pépin, T. "Sur la formule $2^{2^n} + 1$." *Comptes rendus de l'Académie des Sciences*, 85 (1877): 329–331.

[13] Rabinowitz, G. "Eindeutigkeit der Zerlegung in Primzahlfaktoren in quadratischen Zahlkörpern." *Proceedings of the Fifth International Congress of Mathematicians*, vol. 1, pp. 418–421. Cambridge, 1913.

[14] Ribenboim, P. *The New Book of Prime Number Records*, 3rd ed. Springer, 1996.

[15] Stark, H.M. "A complete determination of the complex quadratic fields of class-number one." *Michigan Mathematical Journal*, 14 (1967): 1–27.

[16] Tao, T. "Almost all orbits of the Collatz map attain almost bounded values." *Forum of Mathematics, Pi*, 10 (2022): e12.

[17] Terras, R. "A stopping time problem on the positive integers." *Acta Arithmetica*, 30 (1976): 241–252.

A Fermat Number Factoring Data

n	F_n	Status	Smallest prime factor
0	3	Prime	—
1	5	Prime	—
2	17	Prime	—
3	257	Prime	—
4	65,537	Prime	—
5	4,294,967,297	Composite	641
6	$\approx 1.84 \times 10^{19}$	Composite	274,177
7	$\approx 3.40 \times 10^{38}$	Composite	59,649,589,127,497,217
8	$\approx 1.16 \times 10^{77}$	Composite	1,238,926,361,552,897

Table 1: Fermat numbers F_0 through F_8

B Euler's Formula: Complete Table of Values

n	$P(n)$	Prime?	n	$P(n)$
0	41	Yes	20	461
1	43	Yes	21	503
2	47	Yes	22	547
3	53	Yes	23	593
4	61	Yes	24	641
5	71	Yes	25	691
6	83	Yes	26	743
7	97	Yes	27	797
8	113	Yes	28	853
9	131	Yes	29	911
10	151	Yes	30	971
11	173	Yes	31	1033
12	197	Yes	32	1097
13	223	Yes	33	1163
14	251	Yes	34	1231
15	281	Yes	35	1301
16	313	Yes	36	1373
17	347	Yes	37	1447
18	383	Yes	38	1523
19	421	Yes	39	1601

Table 2: All 40 prime values of $P(n) = n^2 + n + 41$ for $n = 0, \ldots, 39$. All entries are prime.

C Collatz Sequences for Selected Starting Values

Example C.1 (Collatz sequence for $n = 27$). The sequence beginning at 27 requires 111 steps and reaches a maximum of 9,232:

$$27 \to 82 \to 41 \to 124 \to 62 \to 31 \to 94 \to 47 \to 142 \to 71 \to 214 \to 107 \to 322 \to \cdots$$

$$\cdots \to 9232 \to 4616 \to 2308 \to 1154 \to 577 \to \cdots \to 4 \to 2 \to 1.$$

Example C.2 (Total stopping times). The total stopping times show enormous variability:

n	Steps	Max value	n
1	0	1	50
3	7	16	100
7	16	52	200
15	17	160	500
27	111	9,232	1,000

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