Elements of Integral Equations

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)\,dt$$

A Comprehensive Introduction

GAURAV TIWARI

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"Since a general solution must be judged impossible from want of analysis, we must be content with the knowledge of some special cases, and that all the more, since the development of various cases seems to be the only way of bringing us at last to a more perfect knowledge."

— Leonhard Euler

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Contents

Abstract

Integral equations arise naturally in many branches of mathematics, physics, and engineering. They provide an alternative—and often superior—formulation of problems originally expressed as differential equations. This monograph presents a self-contained introduction to the theory of linear integral equations.

We begin with the fundamental definitions and classification of integral equations into Fredholm and Volterra types, and discuss the various kinds within each family. We then develop the function-analytic tools needed for their study: square integrable functions, inner products, norms, and the associated inequalities of Schwarz and Minkowski. The trial method for verifying solutions is presented with detailed examples. The interplay between differential and integral equations is explored in depth: we show how initial value problems yield Volterra equations and boundary value problems yield Fredholm equations, and demonstrate the reverse conversion by systematic differentiation using the Leibniz rule.

Throughout, we emphasize worked examples, explicit computations, and the geometric intuition behind the theory.

Keywords: Integral equations, Fredholm equations, Volterra equations, kernels, square integrable functions, norms, Leibniz rule, Green's functions

1 Definitions and Types of Integral Equations

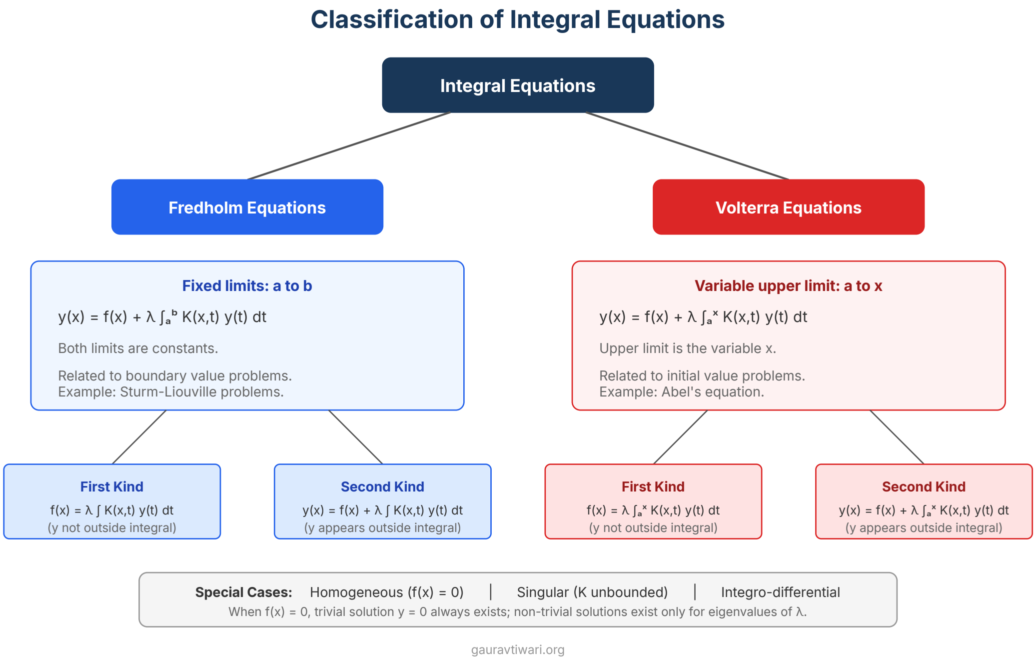

Figure 1: Classification of integral equations into Fredholm and Volterra types, with their first and second kind sub-classifications

1.1 What is an Integral Equation?

Definition 1.1 (Integral Equation). An integral equation is an equation in which an unknown function $y(x)$ appears under one or more integration signs. In its most general linear form it reads

$$g(x)\,y(x) = f(x) + \lambda \int_a^{\square} K(x,t)\,y(t)\,dt, \tag{1}$$

where g, f, and K are given (known) functions, λ is a non-zero complex parameter, and the symbol $\square$ stands for either a fixed constant b or the variable x.

The function $K(x,t)$ is called the kernel of the integral equation. The nature of K profoundly influences the existence, uniqueness, and structure of solutions. We shall return to kernels in Section 1.7.

Remark 1.1. Any integral calculus statement such as $y = \int_a^b \phi(x)\,dx$ can be viewed as a (trivial) integral equation. The theory becomes non-trivial when the unknown function appears *inside* the integral.

1.2 Linear and Non-linear Integral Equations

Definition 1.2 (Linearity). The integral equation (1) is called linear because only linear operations—multiplication by known functions and integration—are performed on the unknown y. An integral equation that involves y^2, $\sin y$, or any other non-linear operation on y is called non-linear.

Example 1.1. The equation

$$y(x) = x + \int_0^1 \sin\bigl(y(t)\bigr)\,dt$$

is *non-linear* because the unknown y appears inside the sine function.

Throughout this monograph, unless stated otherwise, "integral equation" means *linear* integral equation.

1.3 Fredholm and Volterra Types

The classification depends on the upper limit of integration in (1).

Definition 1.3 (Fredholm Integral Equation). When both limits of integration are constants, i.e. $\square = b$, the equation

$$g(x)\,y(x) = f(x) + \lambda \int_a^b K(x,t)\,y(t)\,dt \tag{2}$$

is called a Fredholm integral equation.

Definition 1.4 (Volterra Integral Equation). When the upper limit is the variable x, i.e. $\square = x$, the equation

$$g(x)\,y(x) = f(x) + \lambda \int_a^x K(x,t)\,y(t)\,dt \tag{3}$$

is called a Volterra integral equation.

Remark 1.2. One can view a Volterra equation as a special case of a Fredholm equation with $K(x,t) = 0$ for $t > x$. This perspective is useful in the abstract theory but obscures the very different analytic character of the two types.

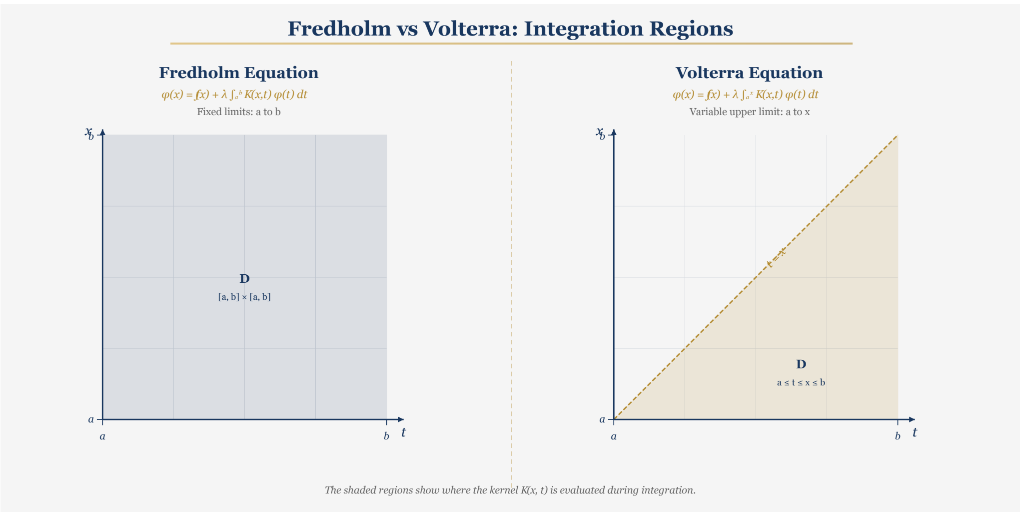

Figure 2: Comparison of Fredholm and Volterra integral equations: fixed vs. variable upper limits of integration and their distinct analytic character

1.4 Kinds of Fredholm Equations

Depending on the values of $g(x)$ and $f(x)$ in (2), we distinguish:

Definition 1.5 (Fredholm Equation of the First Kind). When $g(x) = 0$:

$$f(x) + \lambda \int_a^b K(x,t)\,y(t)\,dt = 0. \tag{4}$$

Here the unknown y appears *only* under the integral sign.

Definition 1.6 (Fredholm Equation of the Second Kind). When $g(x) = 1$:

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\, y(t)\, dt. \tag{5}$$

Definition 1.7 (Homogeneous Fredholm Equation of the Second Kind). When $g(x) = 1$ and $f(x) = 0$:

$$y(x) = \lambda \int_a^b K(x,t)\, y(t)\, dt. \tag{6}$$

Definition 1.8 (Fredholm Equation of the Third Kind). The general equation (2) with $f(x) \neq 0$ and $1 \neq g(x) \neq 0$ is sometimes called a Fredholm equation of the third or final kind.

1.5 Kinds of Volterra Equations

The Volterra classification mirrors the Fredholm one.

Definition 1.9 (Volterra Equation of the First Kind). When $g(x) = 0$:

$$f(x) + \lambda \int_a^x K(x,t)\, y(t)\, dt = 0. \tag{7}$$

Definition 1.10 (Volterra Equation of the Second Kind). When $g(x) = 1$:

$$y(x) = f(x) + \lambda \int_a^x K(x,t)\, y(t)\, dt. \tag{8}$$

Definition 1.11 (Homogeneous Volterra Equation of the Second Kind). When $g(x) = 1$ and $f(x) = 0$:

$$y(x) = \lambda \int_a^x K(x,t)\, y(t)\, dt. \tag{9}$$

Classification Summary.

Type	Upper limit	$g(x)$	$f(x)$
Fredholm, 1st kind	b (constant)	0	given
Fredholm, 2nd kind	b (constant)	1	given
Fredholm, homogeneous	b (constant)	1	0
Volterra, 1st kind	x (variable)	0	given
Volterra, 2nd kind	x (variable)	1	given
Volterra, homogeneous	x (variable)	1	0

1.6 Singular Integral Equations

Definition 1.12 (Singular Integral Equation). An integral equation is called singular if either:

(i) one or both limits of integration are infinite, or

(ii) the kernel $K(x,t)$ becomes unbounded at one or more points in $[a, \square]$.

Example 1.2 (Type I: Infinite limits).

$$y(x) = 3x^2 + \lambda \int_{-\infty}^{\infty} e^{-|x-t|}\, y(t)\, dt.$$

This is singular because both limits are infinite.

Example 1.3 (Type II: Unbounded kernel).

$$y(x) = f(x) + \int_0^x \frac{1}{(x-t)^n}\, y(t)\, dt, \quad 0 < n < 1.$$

The kernel $K(x,t) = (x-t)^{-n}$ is unbounded as $t \to x^-$. This is called an Abel-type integral equation and arises in many physical problems including the tautochrone problem.

Remark 1.3. The Abel integral equation

$$f(x) = \int_0^x \frac{y(t)}{\sqrt{x-t}}\, dt$$

has the elegant closed-form solution

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)}{\sqrt{x-t}}\, dt.$$

This was one of the earliest integral equations to be solved, by Niels Henrik Abel in 1823.

1.7 Kernels

The kernel $K(x,t)$ encodes the structure of the integral equation. Several special types are of particular importance.

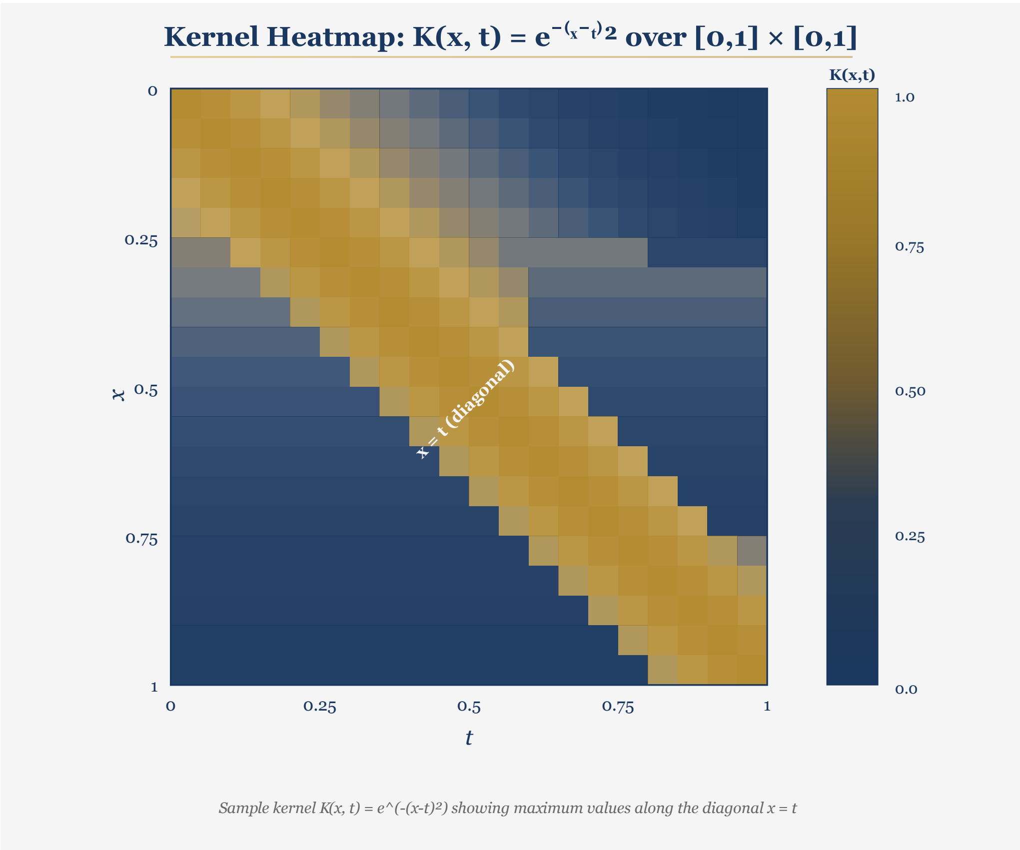

Figure 3: Heatmap visualization of different kernel types: symmetric, separable, and difference kernels exhibit distinctive structural patterns in the (x,t)-plane

Definition 1.13 (Symmetric (Hermitian) Kernel). A kernel is symmetric (or Hermitian) if

$$K(x,t) = \overline{K(t,x)}$$

for all x, t in the domain. If K is real-valued, this reduces to $K(x,t) = K(t,x)$.

Example 1.4. $K(x,t) = \sin(x+t)$ is symmetric since $\sin(x+t) = \sin(t+x)$.

Example 1.5. $K(x,t) = x - t$ is *anti-symmetric*: $K(x,t) = -K(t,x)$.

Definition 1.14 (Separable (Degenerate) Kernel). A kernel is separable or degenerate if it can be written as a finite sum

$$K(x,t) = \sum_{i=1}^{n} \phi_i(x)\,\psi_i(t) \tag{10}$$

where each ϕ_i depends only on x and each ψ_i depends only on t.

Remark 1.4. Integral equations with separable kernels can be reduced to systems of linear algebraic equations. This makes them one of the most tractable classes. For instance, if $K(x,t) = \sum_{i=1}^{n} \phi_i(x)\psi_i(t)$, then the Fredholm equation of the second kind reduces to an $n \times n$ linear system for the coefficients $c_i = \int_a^b \psi_i(t)\,y(t)\,dt$.

Definition 1.15 (Difference Kernel). A kernel of the form $K(x,t) = K(x-t)$ is called a difference kernel.

Definition 1.16 (Resolvent (Reciprocal) Kernel). If the solution of

$$y(x) = f(x) + \lambda \int_a^{\square} K(x,t)\,y(t)\,dt$$

can be written as

$$y(x) = f(x) + \lambda \int_a^{\square} \mathfrak{R}(x,t;\lambda)\,f(t)\,dt,$$

then $\mathfrak{R}(x,t;\lambda)$ is called the resolvent kernel (or reciprocal kernel).

Key Insight. The resolvent kernel $\mathfrak{R}$ transforms the problem from "solving for y given f" into a direct integral transform of f. Finding $\mathfrak{R}$ is equivalent to solving the integral equation completely.

1.8 Integral Equations of Convolution Type

Definition 1.17 (Convolution). Let $y_1(x)$ and $y_2(x)$ be continuous functions. Their convolution is

$$(y_1 * y_2)(x) = \int_{-\infty}^{\infty} y_1(x-t)\,y_2(t)\,dt = \int_{-\infty}^{\infty} y_2(x-t)\,y_1(t)\,dt. \tag{11}$$

When the kernel is a difference kernel $K(x,t) = K(x-t)$, the integral equation is said to be of convolution type. Such equations can often be solved using the Fourier or Laplace transform, since convolution in the spatial domain corresponds to multiplication in the transform domain.

Theorem 1.1 (Convolution Theorem for Laplace Transforms). If $\mathcal{L}\{y_1\} = Y_1(s)$ and $\mathcal{L}\{y_2\} = Y_2(s)$, then

$$\mathcal{L}\{y_1 * y_2\}(s) = Y_1(s) \cdot Y_2(s).$$

Example 1.6. Consider the Volterra equation of convolution type:

$$y(x) = f(x) + \lambda \int_0^x e^{-(x-t)}\, y(t)\, dt.$$

Taking Laplace transforms: $Y(s) = F(s) + \lambda \cdot \frac{1}{s+1} \cdot Y(s)$, which gives $Y(s) = \frac{F(s)(s+1)}{s+1-\lambda}$. The solution is obtained by inverse Laplace transform.

1.9 Eigenvalues and Eigenfunctions

Definition 1.18 (Eigenvalue and Eigenfunction). Consider the homogeneous integral equation

$$y(x) = \lambda \int_a^{\square} K(x,t)\, y(t)\, dt. \tag{12}$$

The trivial solution $y(x) \equiv 0$ always satisfies this equation. A value λ_0 for which a non-trivial solution $y_0(x) \not\equiv 0$ exists is called an eigenvalue. The corresponding non-trivial solution y_0 is called an eigenfunction.

Remark 1.5. Key properties of eigenvalues and eigenfunctions:

(a) By convention, $\lambda = 0$ is excluded from consideration (since every function would be an “eigenfunction” for $\lambda = 0$).

(b) If $y(x)$ is an eigenfunction for λ_0, then $c \cdot y(x)$ is also an eigenfunction for any non-zero constant c. That is, eigenfunctions are determined only up to a scalar multiple.

(c) For symmetric kernels, all eigenvalues are real.

(d) For symmetric kernels, eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Example 1.7. Consider the homogeneous equation

$$y(x) = \lambda \int_0^{2\pi} \sin(x+t)\, y(t)\, dt.$$

Using the addition formula $\sin(x+t) = \sin x \cos t + \cos x \sin t$, this is a separable kernel with $n = 2$. Setting $A = \int_0^{2\pi} \cos t\, y(t)\, dt$ and $B = \int_0^{2\pi} \sin t\, y(t)\, dt$, we get $y(x) = \lambda(A \sin x + B \cos x)$. Substituting back yields the system $A = \lambda\pi B$ and $B = \lambda\pi A$, giving $\lambda^2\pi^2 = 1$, so $\lambda = \pm 1/\pi$. The eigenfunctions are $y(x) = \sin x + \cos x$ (for $\lambda = 1/\pi$) and $y(x) = \sin x - \cos x$ (for $\lambda = -1/\pi$).

1.10 The Leibniz Rule

The Leibniz rule for differentiation under the integral sign is an essential tool throughout this monograph.

Theorem 1.2 (Leibniz Rule). Let $F(x,t)$ and $\partial F/\partial x$ be continuous, and let $G(x)$ and $H(x)$ be continuously differentiable. Then

$$\frac{d}{dx} \int_{G(x)}^{H(x)} F(x,t)\, dt = \int_{G(x)}^{H(x)} \frac{\partial F}{\partial x}\, dt + F\big(x, H(x)\big)\, H'(x) - F\big(x, G(x)\big)\, G'(x). \tag{13}$$

Corollary 1.3. When the limits are constants ($G(x) = a$, $H(x) = b$):

$$\frac{d}{dx} \int_a^b F(x,t)\, dt = \int_a^b \frac{\partial F}{\partial x}\, dt.$$

Corollary 1.4 (Fundamental Theorem of Calculus). When $F(x,t) = f(t)$ is independent of x and the limits are a and x:

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x).$$

1.11 Reduction of Multiple Integrals to Simple Integrals

Iterated integrals arise naturally when integrating differential equations multiple times. The following result converts them to single integrals.

Theorem 1.5 (Repeated Integral Formula). For a continuous function f and integer $n \geq 1$:

$$\underbrace{\int_a^t \int_a^{s_{n-1}} \cdots \int_a^{s_1}}_{n \text{ integrals}} f(x)\,dx\,ds_1 \cdots ds_{n-1} = \frac{1}{(n-1)!} \int_a^t (t-x)^{n-1} f(x)\,dx. \tag{14}$$

Proof. We prove this by induction on n.

Base case ($n = 1$): Both sides reduce to $\int_a^t f(x)\,dx$. ?

Inductive step: Assume the formula holds for $n-1$. Consider the n-fold integral

$$I_n = \int_a^t \left[\frac{1}{(n-2)!} \int_a^s (s-x)^{n-2} f(x)\,dx \right] ds.$$

Interchanging the order of integration (justified by Fubini's theorem since f is continuous):

$$I_n = \frac{1}{(n-2)!} \int_a^t f(x) \left[\int_x^t (s-x)^{n-2}\,ds \right] dx = \frac{1}{(n-2)!} \int_a^t f(x) \cdot \frac{(t-x)^{n-1}}{n-1}\,dx = \frac{1}{(n-1)!} \int_a^t (t-x)^{n-1} f(x)\,dx.$$

□

Example 1.8. Evaluate $\int_0^1 x^2\,dx^2$ (a double integral with $f(x) = x^2$, $n = 2$, $a = 0$, $t = 1$).

Solution. By the repeated integral formula:

$$\int_0^1 x^2\,dx^2 = \frac{1}{1!} \int_0^1 (1-x)\,x^2\,dx = \int_0^1 (x^2 - x^3)\,dx = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Example 1.9. Evaluate the triple integral $\int_0^1 e^x\,dx^3$.

Solution.

$$\int_0^1 e^x\,dx^3 = \frac{1}{2!} \int_0^1 (1-x)^2\,e^x\,dx = \frac{1}{2}\big[(1-x)^2 e^x + 2(1-x)e^x - 2e^x\big]_0^1 = \frac{1}{2}\big[-2e + 0 + 2\big] = 1 - e \approx -1.718.$$

(Integration by parts applied twice.)

2 Square Integrable Functions, Norms, and the Trial Method

2.1 Square Integrable Functions

The natural function space for integral equation theory is the space of square integrable functions, which provides the geometric framework of inner products, norms, and orthogonality.

Definition 2.1 (Square Integrable Function). A function $y(x)$ is said to be square integrable (or an $\mathfrak{L}_2$-function) on the interval (a, b) if

$$\int_a^b |y(x)|^2\,dx < \infty. \tag{15}$$

Equivalently, $\int_a^b y(x)\,\overline{y(x)}\,dx < \infty$. Such a function is also called a regular function.

Example 2.1. The function $y(x) = 1/\sqrt{x}$ on $(0, 1)$: we compute $\int_0^1 |y|^2\, dx = \int_0^1 x^{-1}\, dx = \lim_{\varepsilon\to 0^+}[\ln x]_\varepsilon^1 = +\infty$. Hence $1/\sqrt{x} \notin \mathfrak{L}_2(0, 1)$.

Example 2.2. The function $y(x) = x^{-1/3}$ on $(0, 1)$: $\int_0^1 x^{-2/3}\, dx = [3x^{1/3}]_0^1 = 3 < \infty$. Hence $x^{-1/3} \in \mathfrak{L}_2(0, 1)$, even though it is unbounded.

Definition 2.2 ($\mathfrak{L}_2$-Kernel). The kernel $K(x, t)$, viewed as a function of two variables, is an $\mathfrak{L}_2$-kernel if

$$\int_a^b \int_a^b |K(x, t)|^2\, dx\, dt < \infty. \tag{16}$$

2.2 Inner Product and Orthogonality

Definition 2.3 (Inner Product). The inner product of two complex $\mathfrak{L}_2$-functions ϕ and ψ on $[a, b]$ is defined as

$$(\phi, \psi) = \int_a^b \phi(x)\, \overline{\psi(x)}\, dx. \tag{17}$$

Definition 2.4 (Orthogonality). Two $\mathfrak{L}_2$-functions ϕ and ψ are orthogonal on $[a, b]$ if $(\phi, \psi) = 0$.

Example 2.3. The functions $\phi(x) = \sin(nx)$ and $\psi(x) = \sin(mx)$ with $m \neq n$ (both positive integers) are orthogonal on $[0, \pi]$:

$$\int_0^\pi \sin(nx) \sin(mx)\, dx = 0.$$

This is the fundamental orthogonality relation underlying Fourier sine series.

Remark 2.1. The inner product (17) makes $\mathfrak{L}_2(a, b)$ into a Hilbert space. This is the natural setting for the spectral theory of integral operators.

2.3 Norms and Inequalities

Definition 2.5 (Norm). The norm of a complex function $y(x)$ on $[a, b]$ is

$$\|y\| = \sqrt{(y, y)} = \sqrt{\int_a^b |y(x)|^2\, dx}\,. \tag{18}$$

Theorem 2.1 (Cauchy–Schwarz Inequality). For any $\phi, \psi \in \mathfrak{L}_2(a, b)$:

$$|(\phi, \psi)| \leq \|\phi\| \cdot \|\psi\|. \tag{19}$$

Equivalently, $\left|\int_a^b \phi(x)\overline{\psi(x)}\, dx\right|^2 \leq \int_a^b |\phi|^2\, dx \cdot \int_a^b |\psi|^2\, dx$.

Proof. For any $\lambda \in \mathbb{C}$, we have $\|\phi + \lambda\psi\|^2 \geq 0$. Expanding:

$$\|\phi\|^2 + \lambda\, (\psi, \phi) + \bar{\lambda}\, (\phi, \psi) + |\lambda|^2 \|\psi\|^2 \geq 0.$$

Choose $\lambda = -(\phi, \psi)/\|\psi\|^2$ (assuming $\psi \neq 0$). Substituting and simplifying yields

$$\|\phi\|^2 - \frac{|(\phi, \psi)|^2}{\|\psi\|^2} \geq 0,$$

which gives $|(\phi, \psi)|^2 \leq \|\phi\|^2 \|\psi\|^2$. □

Theorem 2.2 (Minkowski's (Triangle) Inequality). For any $\phi, \psi \in \mathfrak{L}_2(a, b)$:

$$\|\phi + \psi\| \leq \|\phi\| + \|\psi\|. \tag{20}$$

Proof.

$$\begin{aligned}
\|\phi+\psi\|^2 &= (\phi+\psi,\phi+\psi) \\
&= \|\phi\|^2 + (\phi,\psi) + (\psi,\phi) + \|\psi\|^2 \\
&= \|\phi\|^2 + 2\,\mathrm{Re}(\phi,\psi) + \|\psi\|^2 \\
&\le \|\phi\|^2 + 2|(\phi,\psi)| + \|\psi\|^2 \\
&\le \|\phi\|^2 + 2\|\phi\|\|\psi\| + \|\psi\|^2 \quad \text{(by Cauchy–Schwarz)} \\
&= (\|\phi\| + \|\psi\|)^2.
\end{aligned}$$

Taking square roots gives the result. □

Geometric Interpretation. The Cauchy–Schwarz inequality says that the "angle" between two functions is well-defined. The triangle inequality says that the shortest path between two "points" in function space is the "straight line." These are the same properties that hold for vectors in $\mathbb{R}^n$—the space $\mathfrak{L}_2$ is an infinite-dimensional generalization of Euclidean space.

2.4 The Trial Method (Verification of Solutions)

The trial method consists of substituting a candidate solution into the integral equation and verifying that it satisfies the equation identically.

Example 2.4. Show that $y(x) = (1+x^2)^{-3/2}$ is a solution of the Volterra equation

$$y(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2}\, y(t)\, dt. \tag{21}$$

Solution. We substitute $y(t) = (1+t^2)^{-3/2}$ into the right-hand side:

$$\begin{aligned}
\text{RHS} &= \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2}\cdot(1+t^2)^{-3/2}\, dt \\
&= \frac{1}{1+x^2} - \frac{1}{1+x^2}\int_0^x \frac{t}{(1+t^2)^{3/2}}\, dt.
\end{aligned}\tag{22}$$

The factor $\frac{1}{1+x^2}$ is independent of t and can be taken outside the integral. Now evaluate the inner integral via the substitution $u = 1+t^2$, $du = 2t\,dt$:

$$\int_0^x \frac{t}{(1+t^2)^{3/2}}\, dt = \frac{1}{2}\int_1^{1+x^2} u^{-3/2}\, du = \frac{1}{2}\left[-2u^{-1/2}\right]_1^{1+x^2} = 1 - \frac{1}{\sqrt{1+x^2}}. \tag{23}$$

Substituting (23) into (22):

$$\begin{aligned}
\text{RHS} &= \frac{1}{1+x^2} - \frac{1}{1+x^2}\left(1 - \frac{1}{\sqrt{1+x^2}}\right) \\
&= \frac{1}{1+x^2}\cdot\frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-3/2} = y(x).
\end{aligned}\tag{24}$$

Example 2.5. Show that $y(x) = e^x$ is a solution of

$$y(x) = 1 + \int_0^x y(t)\, dt.$$

Solution. Substituting $y(t) = e^t$:

$$\text{RHS} = 1 + \int_0^x e^t\, dt = 1 + [e^t]_0^x = 1 + e^x - 1 = e^x = y(x).$$

Example 2.6. Show that $y(x) = \cosh x$ satisfies

$$y(x) = 1 + \int_0^x (x - t)\, y(t)\, dt.$$

Solution. Substitute $y(t) = \cosh t$:

$$\begin{aligned} \text{RHS} &= 1 + \int_0^x (x - t) \cosh t\, dt \\ &= 1 + x \int_0^x \cosh t\, dt - \int_0^x t \cosh t\, dt \\ &= 1 + x \sinh x - [t \sinh t - \cosh t]_0^x \\ &= 1 + x \sinh x - x \sinh x + \cosh x - 1 = \cosh x = y(x). \end{aligned}$$

Remark 2.2. The trial method is not a solution technique per se—it requires you to already know (or guess) the answer. Its pedagogical value lies in building familiarity with the manipulation of integral equations. In practice, systematic methods such as successive approximations (Neumann series), resolvent kernels, and transform methods are used to *find* solutions.

2.5 The Neumann Series: A Preview

While a full treatment of solution methods is beyond our present scope, we mention the Neumann series as motivation.

Theorem 2.3 (Neumann Series). For the Fredholm equation of the second kind $y = f + \lambda K y$ (where K denotes the integral operator), if $|\lambda| \cdot \|K\| < 1$, the unique solution is given by

$$y = \sum_{n=0}^{\infty} \lambda^n K^n f = f + \lambda K f + \lambda^2 K^2 f + \cdots \tag{25}$$

where $K^n f$ denotes n successive applications of the integral operator.

Remark 2.3. The Neumann series is the integral-equation analogue of the geometric series $(1 - r)^{-1} = 1 + r + r^2 + \cdots$ for $|r| < 1$. The condition $|\lambda| \cdot \|K\| < 1$ is analogous to $|r| < 1$.

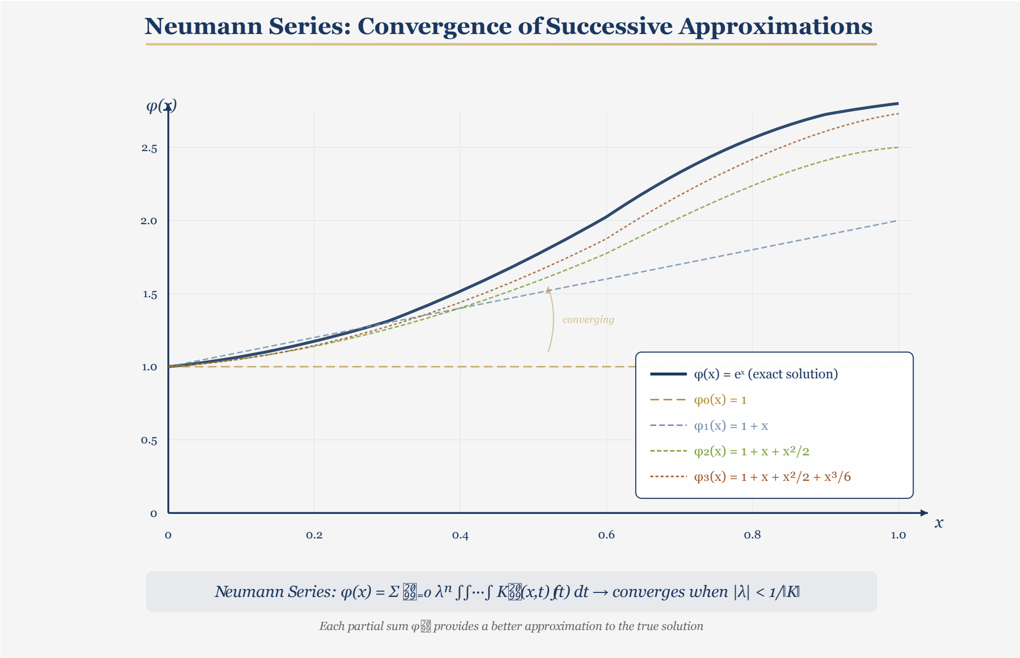

Figure 4: Convergence of the Neumann series: successive partial sums $\sum_{n=0}^{N} \lambda^n K^n f$ approach the exact solution as $N \to \infty$ when $|\lambda| \cdot \|K\| < 1$

3 Converting Differential Equations into Integral Equations

3.1 Motivation

Differential equations and integral equations are two representations of the same underlying mathematical problem. Converting between them is valuable for several reasons:

(i) Different solution techniques: Some equations that resist standard ODE methods become tractable as integral equations (and vice versa).

(ii) Numerical stability: Integral equations often enjoy better numerical conditioning than their differential counterparts.

(iii) Existence and uniqueness: Proving the existence and uniqueness of solutions is frequently easier in integral form via fixed-point theorems.

(iv) Built-in boundary conditions: The integral equation incorporates the boundary/initial conditions directly, so they cannot be accidentally neglected.

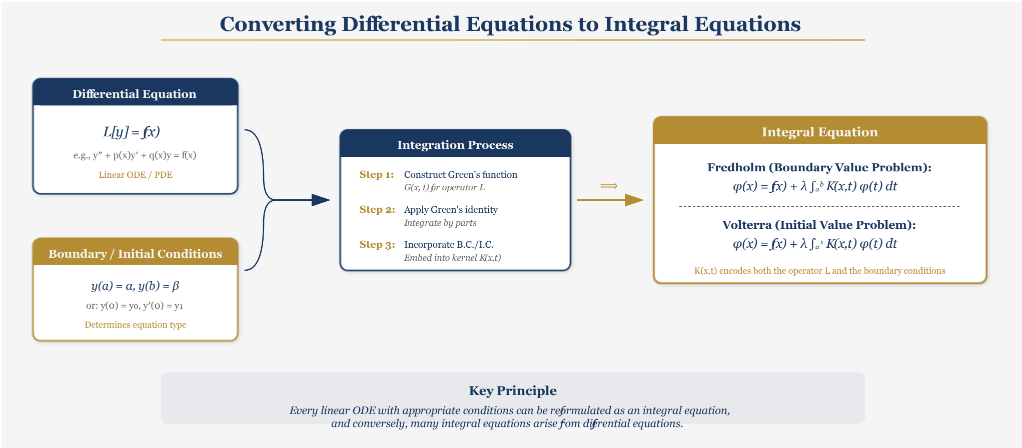

Figure 5: Schematic of the conversion process from differential equations to integral equations: integration absorbs derivatives while boundary/initial conditions are encoded in the free term

The Central Correspondence.

ODE Type	Integral Equation Type	Upper Limit
Initial Value Problem (IVP)	Volterra	Variable x
Boundary Value Problem (BVP)	Fredholm	Constant b

3.2 First-Order Equations

Example 3.1. Convert $\dfrac{dy}{dx} + 5y + 1 = 0$ with $y(0) = 1$ into an integral equation.

Solution. Integrate from 0 to x:

$$\int_0^x y'(t)\, dt + 5 \int_0^x y(t)\, dt + \int_0^x 1\, dt = 0.$$

The first integral gives $y(x) - y(0) = y(x) - 1$. Hence

$$y(x) - 1 + 5 \int_0^x y(t)\, dt + x = 0,$$

which rearranges to

$$y(x) = (1 - x) - 5 \int_0^x y(t)\, dt. \tag{26}$$

This is a Volterra equation of the second kind with kernel $K(x,t) = -5$ and $f(x) = 1 - x$.

Remark 3.1. Notice that the initial condition $y(0) = 1$ has been absorbed into the free term $f(x) = 1 - x$. This illustrates the general principle: integral equations *encode* boundary conditions.

3.3 Second-Order Initial Value Problems

Definition 3.1 (Initial Value Problem). A second-order ODE with both conditions specified at the same point:

$$y'' = F(x, y, y'), \quad y(a) = \alpha, \quad y'(a) = \beta$$

is called an initial value problem (IVP).

3.3.1 General Procedure

Given $y''(x) = h(x, y)$ with $y(a) = \alpha$ and $y'(a) = \beta$:

Step 1. Integrate from a to x:

$$y'(x) - y'(a) = \int_a^x h(t, y(t))\,dt \implies y'(x) = \beta + \int_a^x h(t, y(t))\,dt.$$

Step 2. Integrate again from a to x:

$$y(x) - y(a) = \beta(x - a) + \int_a^x \int_a^s h(t, y(t))\,dt\,ds.$$

Step 3. Apply the repeated integral formula (Theorem 1.5) to convert the double integral:

$$y(x) = \alpha + \beta(x - a) + \int_a^x (x - t)\,h(t, y(t))\,dt. \tag{27}$$

This is a Volterra integral equation (of the second kind if h is linear in y).

Example 3.2. Convert $y'' + y = 0$ with $y(0) = 0$, $y'(0) = 0$ to an integral equation.

Solution. Here $h(x, y) = -y(x)$, $a = 0$, $\alpha = 0$, $\beta = 0$.

Step 1. Integrate once: $y'(x) - 0 = -\int_0^x y(t)\,dt$, so $y'(x) = -\int_0^x y(t)\,dt$.

Step 2. Integrate again: $y(x) - 0 = -\int_0^x \int_0^s y(t)\,dt\,ds$.

Step 3. Apply the repeated integral formula:

$$y(x) = -\int_0^x (x - t)\,y(t)\,dt. \tag{28}$$

This is a homogeneous Volterra equation of the second kind with kernel $K(x, t) = -(x - t)$.

Example 3.3. Convert $y'' - 2y = e^x$ with $y(0) = 1$, $y'(0) = -1$ to an integral equation.

Solution. Rewrite as $y''(x) = 2y(x) + e^x$.

Step 1. Integrate from 0 to x: $y'(x)+1 = \int_0^x [2y(t)+e^t]\,dt$, so $y'(x) = -1+\int_0^x 2y(t)\,dt+(e^x-1)$.

Step 2. Integrate again: $y(x) - 1 = -x + \int_0^x\int_0^s 2y(t)\,dt\,ds + \int_0^x (e^t - 1)\,dt$.

Apply the repeated integral formula to the double integral and evaluate $\int_0^x (e^t-1)\,dt = e^x-1-x$:

$$\begin{aligned} y(x) &= 1 - x + (e^x - 1 - x) + 2\int_0^x (x - t)y(t)\,dt \\ &= e^x - 2x + 2\int_0^x (x - t)\,y(t)\,dt. \end{aligned} \tag{29}$$

This is a Volterra equation of the second kind with $f(x) = e^x - 2x$ and $K(x, t) = 2(x - t)$.

3.4 Second-Order Boundary Value Problems

Definition 3.2 (Boundary Value Problem). A second-order ODE with conditions at two different points:

$$y'' = F(x, y, y'), \quad y(a) = A, \quad y(b) = B$$

is called a boundary value problem (BVP).

The key difficulty is that $y'(a)$ is *unknown*. We introduce it as a parameter c and eliminate it using the condition at $x = b$.

3.4.1 General Procedure for $y'' + \lambda y = 0$, $y(0) = 0$, $y(\ell) = 0$

Step 1. Rewrite as $y'' = -\lambda y$ and integrate from 0 to x:

$$y'(x) - y'(0) = -\lambda \int_0^x y(t)\,dt.$$

Set $y'(0) = c$ (unknown): $y'(x) = c - \lambda \int_0^x y(t)\,dt$.

Step 2. Integrate again from 0 to x:

$$y(x) - y(0) = cx - \lambda \int_0^x (x-t)\,y(t)\,dt.$$

Using $y(0) = 0$:

$$y(x) = cx - \lambda \int_0^x (x-t)\,y(t)\,dt. \tag{30}$$

Step 3. Apply $y(\ell) = 0$ to find c:

$$0 = c\ell - \lambda \int_0^\ell (\ell - t)\,y(t)\,dt \implies c = \frac{\lambda}{\ell} \int_0^\ell (\ell - t)\,y(t)\,dt.$$

Step 4. Substitute back into (30):

$$y(x) = \frac{\lambda x}{\ell} \int_0^\ell (\ell - t)\,y(t)\,dt - \lambda \int_0^x (x-t)\,y(t)\,dt.$$

Step 5. Split the first integral at x and combine:

$$y(x) = \lambda \int_0^\ell K(x,t)\,y(t)\,dt, \tag{31}$$

where the Green's function kernel is

$$K(x,t) = \begin{cases} \dfrac{t(\ell - x)}{\ell}, & 0 \le t \le x, \\[2mm] \dfrac{x(\ell - t)}{\ell}, & x \le t \le \ell. \end{cases} \tag{32}$$

Properties of the Green's Function Kernel.

(a) $K(x,t) = K(t,x)$ — the kernel is symmetric. This reflects the self-adjoint nature of the operator d^2/dx^2.

(b) $K(x,t) \ge 0$ on $[0,\ell] \times [0,\ell]$.

(c) $K(0,t) = K(x,0) = K(\ell,t) = K(x,\ell) = 0$ — the kernel vanishes on the boundary, consistent with the boundary conditions.

(d) K is continuous but has a discontinuous first derivative at $x = t$ (the "kink" in Green's function).

Example 3.4. Convert $y'' + \pi^2 y = 0$ with $y(0) = 0$, $y(1) = 0$ to a Fredholm equation.
Solution. Using (31) with $\lambda = \pi^2$ and $\ell = 1$:

$$y(x) = \pi^2 \int_0^1 K(x,t)\, y(t)\, dt$$

where

$$K(x,t) = \begin{cases} t(1-x), & 0 \le t \le x, \\ x(1-t), & x \le t \le 1. \end{cases}$$

The eigenvalues of this kernel are $\lambda_n = n^2\pi^2$ with eigenfunctions $y_n(x) = \sin(n\pi x)$.

3.5 Higher-Order Equations

The method extends naturally. An n-th order ODE requires n integrations and n boundary/initial conditions.

Theorem 3.1. An n-th order linear ODE

$$y^{(n)}(x) = h(x) + \sum_{k=0}^{n-1} p_k(x)\, y^{(k)}(x),$$

with initial conditions $y^{(k)}(a) = \alpha_k$ for $k = 0, 1, \dots, n-1$, can be converted to a Volterra integral equation of the form

$$y(x) = F(x) + \int_a^x K(x,t)\, y(t)\, dt, \tag{33}$$

where F incorporates the initial conditions and the non-homogeneous term h, and K is constructed from the coefficients p_k and powers of $(x - t)$.

Remark 3.2. For constant-coefficient equations, the kernel $K(x,t)$ depends only on $x - t$ (i.e., it is a difference kernel), and Laplace transform methods apply directly.

4 Converting Integral Equations into Differential Equations

4.1 Overview

The process of converting integral equations back into differential equations is the *reverse* of what we did in Section 3. Where we integrated to remove derivatives, we now differentiate to remove integrals.

The Reverse Correspondence.

Integral Equation Type	Produces
Volterra equation	Initial Value Problem
Fredholm equation	Boundary Value Problem

The strategy:

1. Differentiate the integral equation with respect to x using the Leibniz rule (Theorem 1.2).
2. Repeat until the integral term is completely eliminated.
3. Extract boundary/initial conditions by substituting the limits of integration.

4.2 Volterra Equations → Initial Value Problems

Example 4.1. Convert $y(x) = -\int_0^x (x-t)\,y(t)\,dt$ back to a differential equation.
Solution.
Step 1: First differentiation. Apply the Leibniz rule to

$$y(x) = -\int_0^x (x-t)\,y(t)\,dt. \tag{1}$$

Here $F(x,t) = -(x-t)\,y(t)$, $G(x) = 0$, $H(x) = x$.
The Leibniz rule gives:

$$\begin{aligned} y'(x) &= -\int_0^x \frac{\partial}{\partial x}[(x-t)\,y(t)]\,dt - (x-x)\,y(x)\cdot 1 + 0 \\ &= -\int_0^x y(t)\,dt. \end{aligned} \tag{2}$$

Note: $\frac{\partial}{\partial x}(x-t) = 1$, and the boundary term $(x-x)y(x) = 0$.
Step 2: Second differentiation. Differentiate (2):

$$y''(x) = -\frac{d}{dx}\int_0^x y(t)\,dt = -y(x). \tag{3}$$

by the Fundamental Theorem of Calculus.
Hence $y''(x) + y(x) = 0$.
Step 3: Extract initial conditions. Substitute $x = 0$ into (1): $y(0) = -\int_0^0 \cdots = 0$.
Substitute $x = 0$ into (2): $y'(0) = -\int_0^0 \cdots = 0$.
Result: $y'' + y = 0$, $y(0) = 0$, $y'(0) = 0$.

This recovers exactly the IVP from Example 3.2, confirming the round-trip.

Example 4.2. Convert $y(x) = e^x + \int_0^x (x-t)\,y(t)\,dt$ to a differential equation.
Solution.
Differentiate once:

$$y'(x) = e^x + \int_0^x y(t)\,dt. \tag{1}$$

Differentiate again:

$$y''(x) = e^x + y(x). \tag{2}$$

Hence $y'' - y = e^x$.
Initial conditions from the original equation at $x = 0$: $y(0) = e^0 + 0 = 1$ and $y'(0) = e^0 + 0 = 1$.
Result: $y'' - y = e^x$, $y(0) = 1$, $y'(0) = 1$.

Example 4.3. Convert $y(x) = \sin x + 2\int_0^x (x-t)^2\,y(t)\,dt$ to a differential equation.
Solution.
Differentiate once: $y'(x) = \cos x + 4\int_0^x (x-t)\,y(t)\,dt$. (1)
Differentiate again: $y''(x) = -\sin x + 4\int_0^x y(t)\,dt$. (2)
Differentiate a third time: $y'''(x) = -\cos x + 4y(x)$. (3)
Hence $y''' - 4y = -\cos x$.
Conditions at $x = 0$: $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$.
Result: $y''' - 4y = -\cos x$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$.

Remark 4.1. The number of differentiations needed corresponds to the "power" of $(x-t)$ in the kernel plus one. A kernel $(x-t)^{n-1}/(n-1)!$ requires n differentiations to eliminate the integral, producing an n-th order ODE.

4.3 Fredholm Equations → Boundary Value Problems

Example 4.4. Convert to a BVP:

$$y(x) = \lambda \int_0^\ell K(x,t)\, y(t)\, dt$$

where $K(x,t) = \begin{cases} t(\ell - x)/\ell, & 0 < t < x, \\ x(\ell - t)/\ell, & x < t < \ell. \end{cases}$

Solution. Split the integral at x:

$$y(x) = \frac{\lambda(\ell - x)}{\ell} \int_0^x t\, y(t)\, dt + \frac{\lambda x}{\ell} \int_x^\ell (\ell - t)\, y(t)\, dt. \tag{1}$$

First differentiation. Carefully applying the Leibniz rule to each piece:

For the first integral, both the integrand factor $(\ell - x)/\ell$ and the upper limit x depend on x. By the product and Leibniz rules:

$$\frac{d}{dx}\left[\frac{(\ell - x)}{\ell} \int_0^x t\, y(t)\, dt\right] = -\frac{1}{\ell} \int_0^x t\, y(t)\, dt + \frac{(\ell - x)}{\ell} \cdot x\, y(x).$$

For the second integral, x/ℓ and the lower limit x depend on x:

$$\frac{d}{dx}\left[\frac{x}{\ell} \int_x^\ell (\ell - t)\, y(t)\, dt\right] = \frac{1}{\ell} \int_x^\ell (\ell - t)\, y(t)\, dt - \frac{x}{\ell} \cdot (\ell - x)\, y(x).$$

The boundary terms $\frac{(\ell - x)x}{\ell} y(x)$ cancel:

$$y'(x) = -\frac{\lambda}{\ell} \int_0^x t\, y(t)\, dt + \frac{\lambda}{\ell} \int_x^\ell (\ell - t)\, y(t)\, dt. \tag{2}$$

Second differentiation. Differentiate (2):

$$\begin{aligned} y''(x) &= -\frac{\lambda}{\ell} \cdot x\, y(x) - \frac{\lambda}{\ell} \cdot (\ell - x)\, y(x) \\ &= -\frac{\lambda}{\ell}\, y(x)\, [x + \ell - x] = -\lambda\, y(x). \end{aligned} \tag{3}$$

Hence $y'' + \lambda y = 0$.

Boundary conditions. From (1):

- At $x = 0$: $y(0) = 0 + 0 = 0$.
- At $x = \ell$: $y(\ell) = 0 + 0 = 0$.

Result: $y'' + \lambda y = 0$, $y(0) = 0$, $y(\ell) = 0$.

4.4 Practical Guidelines

Theorem 4.1 (Differentiation Count). If an integral equation arose from an n-th order ODE via n integrations, then n differentiations of the integral equation recover the ODE. The order of the resulting ODE equals the number of differentiations needed to eliminate all integrals.

Remark 4.2 (Common Pitfalls). (a) Leibniz rule errors. When x appears both in the limits and in the integrand, all three terms of the Leibniz rule (13) must be accounted for.

(b) Cancellation of boundary terms. In BVP kernels, boundary terms often cancel due to the kernel's continuity at $x = t$. Verify this explicitly.

(c) Extracting conditions. For Volterra equations, substitute $x = a$ (the lower limit) into the equation *and all its derivatives* to obtain initial conditions. For Fredholm equations, substitute both limits $x = a$ and $x = b$ into the original equation.

4.5 The Round-Trip: Verification

Proposition 4.2 (Round-Trip Consistency). If we start with an ODE together with boundary or initial conditions, convert it to an integral equation (Section 3), and then convert back (Section 4), we recover the original ODE and conditions.

This serves as a powerful consistency check. Examples 3.2 and 4.1 demonstrate this round-trip explicitly for the equation $y'' + y = 0$.

Example 4.5 (Full Round-Trip). Start with the BVP: $y'' + 4y = 0$, $y(0) = 0$, $y(\pi) = 0$.
Forward (ODE → IE): Following the procedure of Section 3.4 with $\lambda = 4$ and $\ell = \pi$:

$$y(x) = 4 \int_0^{\pi} K(x,t)\, y(t)\, dt$$

where $K(x,t) = \begin{cases} t(\pi - x)/\pi, & t \le x, \\ x(\pi - t)/\pi, & x \le t. \end{cases}$

Reverse (IE → ODE): Differentiate twice as in Example 4.4: $y'' + 4y = 0$.

Evaluate at boundaries: $y(0) = 0$, $y(\pi) = 0$. □

The eigenvalues are $\lambda_n = n^2$ and the eigenfunctions are $y_n(x) = \sin(nx)$, $n = 1, 2, 3, \ldots$. For $\lambda = 4 = 2^2$, the eigenfunction is $y(x) = \sin(2x)$.

5 Further Topics and Outlook

5.1 Integral Equations and Green's Functions

The piecewise kernel (32) that arose in the BVP conversion is an example of a Green's function. More generally:

Definition 5.1 (Green's Function). The Green's function $G(x,t)$ for a linear differential operator L on $[a,b]$ with homogeneous boundary conditions is the function satisfying

$$L_x\, G(x,t) = \delta(x - t)$$

together with the boundary conditions, where δ is the Dirac delta.

Theorem 5.1. If $G(x,t)$ is the Green's function for L with homogeneous boundary conditions, then the solution of $Ly = f$ with the same boundary conditions is

$$y(x) = \int_a^b G(x,t)\, f(t)\, dt.$$

This shows that solving a BVP is equivalent to finding the Green's function, which in turn is the kernel of the associated Fredholm integral equation.

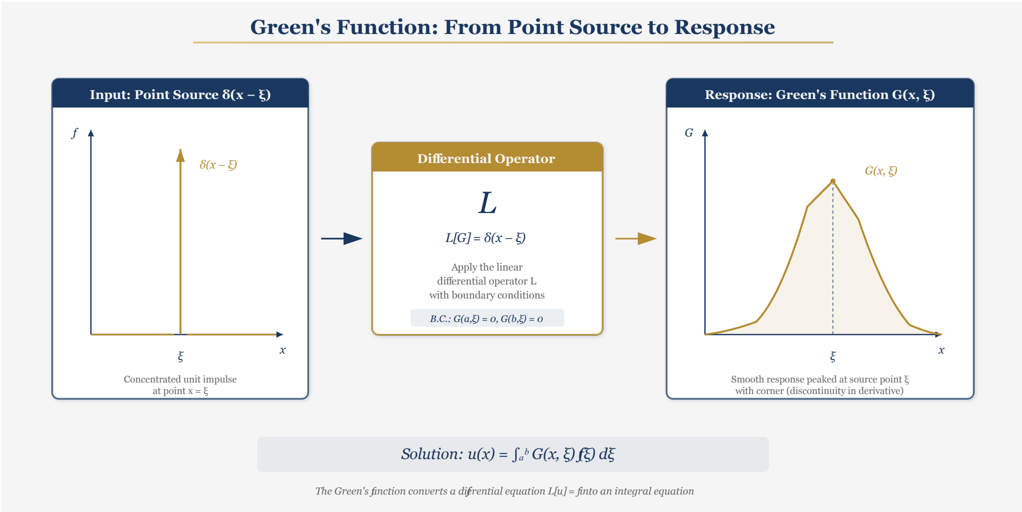

Figure 6: Visualization of the Green's function $G(x,t)$: the characteristic "tent" shape with a ridge along $x = t$ reflects the response of the differential operator to a point source

5.2 Existence and Uniqueness via Integral Equations

One of the great advantages of the integral equation formulation is that it enables clean existence and uniqueness proofs via fixed-point theorems.

Theorem 5.2 (Picard–Lindelöf via Integral Equations). Consider the IVP $y' = f(x,y)$, $y(x_0) = y_0$, where f satisfies a Lipschitz condition in y. The equivalent Volterra integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

defines a contraction mapping on a suitable function space. By the Banach fixed-point theorem, this equation has a unique solution.

Remark 5.1. This is exactly the Picard–Lindelöf theorem, but the proof via integral equations is more transparent than the classical ODE proof: the Picard iterates

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt$$

are simply the successive approximations (Neumann series) applied to the Volterra equation.

5.3 Fredholm Theory: A Brief Synopsis

For the Fredholm equation of the second kind $y = f + \lambda K y$ with $\mathfrak{L}_2$-kernel K:

Theorem 5.3 (Fredholm Alternative). Exactly one of the following holds:

(i) The equation $y = f + \lambda K y$ has a unique solution for every $f \in \mathfrak{L}_2$.

(ii) The homogeneous equation $y = \lambda K y$ has a non-trivial solution (i.e., λ is an eigenvalue).

In case (ii), the non-homogeneous equation has a solution if and only if f is orthogonal to every eigenfunction of the adjoint equation $z = \bar{\lambda} K^* z$.

Remark 5.2. The Fredholm alternative is the infinite-dimensional analogue of the fact that $Ax = b$ has a unique solution if and only if $\det A \neq 0$, and otherwise a solution exists only if b is orthogonal to $\ker(A^T)$.

5.4 Directions for Further Study

The theory of integral equations is vast. We mention several natural continuations:

1. Successive approximations (Neumann series): Iterative construction of solutions converging geometrically.
2. Fredholm determinants: A systematic method for solving Fredholm equations using infinite determinants.
3. Hilbert–Schmidt theory: Spectral decomposition for symmetric $\mathfrak{L}_2$-kernels, analogous to diagonalization of symmetric matrices.
4. Singular integral equations: Cauchy-type and Hilbert-type kernels, with applications to boundary value problems in complex analysis.
5. Non-linear integral equations: Fixed-point methods (Schauder, Leray–Schauder) for equations involving non-linear functions of y.
6. Numerical methods: Quadrature methods, collocation, Galerkin methods, and boundary element methods for computational solutions.
7. Integro-differential equations: Equations involving both derivatives and integrals of the unknown function, arising in viscoelasticity, population dynamics, and control theory.

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