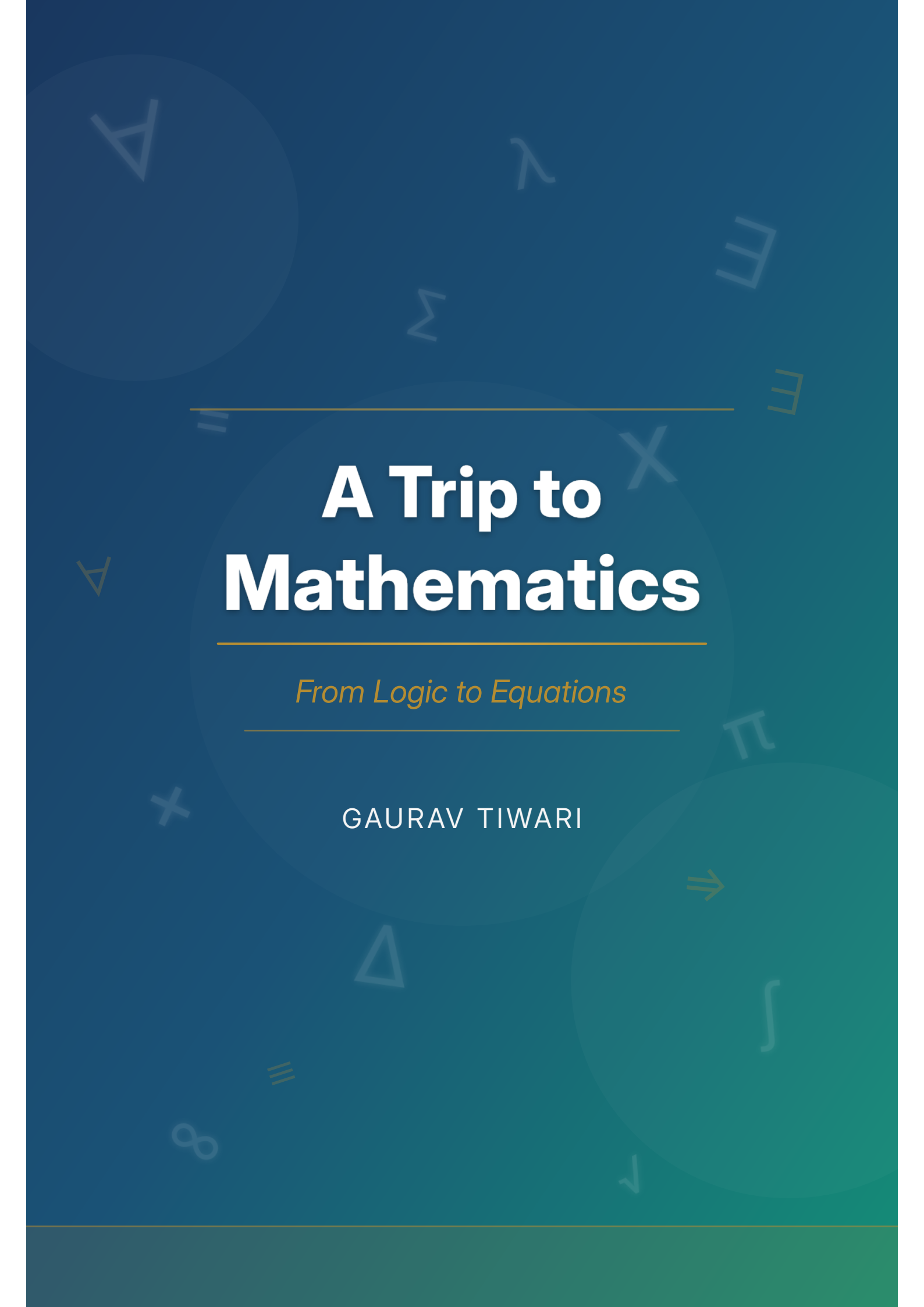


The background is a dark blue to teal gradient. It features several faint, light blue mathematical symbols and shapes scattered across the surface, including a large circle in the top left, a triangle, a lambda symbol, a sigma symbol, a pi symbol, a plus sign, a triangle, an infinity symbol, a square root symbol, and a double arrow. The title is centered and framed by two horizontal lines.

A Trip to Mathematics

From Logic to Equations

GAURAV TIWARI

A Trip to Mathematics: From Logic to Equations

A Beginner's Guide to Mathematical Thinking

Gaurav Tiwari

*“Mathematics is not about numbers, equations, computations, or
algorithms: it is about understanding.”*

— William Paul Thurston

Based on articles from <https://gauravtiwari.org>

Compiled and expanded: February 2026

<https://gauravtiwari.org>

Contents

Preface	5
1 Introduction: Why Mathematics Matters	7
1.1 Mathematics Is Everywhere	7
1.2 The Language of Precision	7
1.3 What You Will Learn in This Book	7
1.4 How Mathematicians Think	8
1.5 A Brief History of Mathematical Thinking	8
2 Mathematical Logic	11
2.1 Sentences and Statements	11
2.2 The Five Logical Connectives	12
2.2.1 Negation (NOT)	12
2.2.2 Conjunction (AND)	12
2.2.3 Disjunction (OR)	13
2.2.4 Implication (IF...THEN)	14
2.2.5 Biconditional (IF AND ONLY IF)	16
2.3 Tautologies, Contradictions, and Contingencies	16
2.4 Logical Equivalences	17
2.5 Quantifiers	17
2.6 Stronger and Weaker Statements	18
2.7 Inductive vs. Deductive Reasoning	18
2.8 Axioms, Postulates, and Proof	18
2.8.1 Common Proof Techniques	19
2.9 Arguments, Premises, and Conclusions	20
3 Numbers and Measurement: Significant Figures	21
3.1 What Are Significant Figures?	21
3.2 The Significance of Significant Figures	22
3.3 Rules for Counting Significant Digits	22
3.4 Scientific Notation	23
3.5 Operations with Significant Figures	23
3.5.1 Addition and Subtraction	23
3.5.2 Multiplication and Division	23
3.6 Rounding Rules	24
3.7 Precision vs. Accuracy	24
3.8 Applications in Science and Engineering	25
3.9 Practice Problems	25

4	Equations: A Basic Introduction	27
4.1	Why Equations Matter	27
4.2	What Exactly Is an Equation?	27
4.3	The Golden Rule of Equations	28
4.4	More Worked Examples	29
4.5	Types of Equations: A Complete Classification	30
4.5.1	Classification by Number of Variables	30
4.5.2	Classification by Degree	30
4.6	Identities vs. Conditional Equations	31
4.7	Solution Sets and Equivalent Equations	31
4.8	The Power of Variables: Why Letters?	31
5	Solving Linear Equations	33
5.1	What Is a Linear Equation?	33
5.2	Forms of Linear Equations	33
5.2.1	Standard Form	33
5.2.2	Slope-Intercept Form	33
5.2.3	Point-Slope Form	33
5.2.4	Intercept Form	34
5.2.5	Special Cases: Horizontal and Vertical Lines	34
5.3	Solving One-Variable Linear Equations	34
5.3.1	Basic Steps	35
5.3.2	Multi-Step Equations	35
5.4	Properties of Equality	36
5.5	Real-World Applications	37
5.6	Systems of Linear Equations	37
5.6.1	Graphical Interpretation	37
5.6.2	Substitution Method	38
5.6.3	Elimination Method	39
5.6.4	Preview: Matrix Methods	40
5.7	Linear Inequalities	40
5.8	Practice Problems	40
6	Beyond Linear: A Preview	43
6.1	Quadratic Equations	43
6.1.1	Solving by Factoring	43
6.1.2	The Quadratic Formula	43
6.1.3	Completing the Square	44
6.2	Polynomial Equations	44
6.3	What Comes Next?	45
	Further Reading	47

Preface

Mathematics is often called the universal language—a system of thought that transcends cultural boundaries and connects every branch of science, engineering, economics, and everyday life. Yet many students approach it with trepidation, unsure where to begin or why it matters.

This mini-book is designed to change that. Starting from the very foundations of *mathematical logic*—the grammar of mathematics—we build steadily toward the practical skill of solving equations. Along the way, we explore how scientists and engineers use *significant figures* to communicate measurements with honesty and precision.

The material is drawn from four interconnected articles originally published on gauravtiwari.org, now substantially expanded with additional examples, practice problems, and explanations suitable for beginners. Whether you are a middle-school student encountering algebra for the first time, a high-school student reviewing fundamentals, or a curious adult returning to mathematics, this book is for you.

Every chapter is self-contained, but they build on one another. Chapter 1 motivates why mathematics matters. Chapter 2 introduces the logical framework that underpins all mathematical reasoning. Chapter 3 covers significant figures—a practical skill for anyone who works with measurements. Chapters 4 and 5 introduce equations and solve them systematically. Chapter 6 offers a preview of what lies beyond.

How to use this book: Read actively. Work through every example with pen and paper. Attempt the exercises before looking at solutions. Mathematics is not a spectator sport—you learn by *doing*.

— Gaurav Tiwari, February 2026

Chapter 1

Introduction: Why Mathematics Matters

1.1 Mathematics Is Everywhere

You might not realize it, but you use mathematics every single day. When you check whether you have enough money to buy lunch, you are doing arithmetic. When you estimate how long it will take to walk to school, you are working with rates and distances. When you decide which phone plan gives you the best deal, you are comparing ratios. Mathematics is not confined to classrooms and textbooks—it is the invisible framework that supports modern life.

Example 1.1 (Mathematics in Daily Life). Consider the simple act of cooking. A recipe says it serves 4 people, but you need to cook for 6. You must multiply every ingredient by $\frac{6}{4} = 1.5$. If the recipe calls for 2 cups of flour, you need $2 \times 1.5 = 3$ cups. This is proportional reasoning—a mathematical skill.

Example 1.2 (Mathematics in Technology). Every time you use a search engine, stream a video, or send a text message, complex mathematical algorithms are at work. Encryption that keeps your online banking secure is based on number theory. The compression that lets you stream music relies on Fourier analysis. The GPS in your phone uses geometry and the theory of relativity.

1.2 The Language of Precision

Natural languages like English are rich and expressive, but they can be ambiguous. Consider the sentence: “I saw the man with the telescope.” Did you use a telescope to see the man, or did you see a man who was holding a telescope? The sentence is ambiguous.

Mathematics eliminates this ambiguity. The statement $2 + 3 = 5$ means exactly one thing, everywhere in the world, regardless of what language you speak. This precision is what makes mathematics so powerful as a tool for science, engineering, and clear thinking.

Definition 1.1 (Mathematics). Mathematics is the study of patterns, structures, quantities, and relationships, expressed through a precise symbolic language governed by logical rules.

1.3 What You Will Learn in This Book

This book takes you on a carefully structured journey through the foundations of mathematical thinking:

1. **Mathematical Logic (Chapter 2):** The grammar of mathematics. You will learn how to construct precise statements, connect them with logical operators, and build valid arguments. This is the foundation everything else rests on.

2. Significant Figures (Chapter 3): How scientists and engineers handle measurements honestly. You will learn to identify meaningful digits, round appropriately, and propagate precision through calculations.
3. Equations: A Basic Introduction (Chapter 4): What equations are, why they matter, and how to classify them. You will understand the difference between expressions and equations, identities and conditional equations, and learn the vocabulary of algebra.
4. Solving Linear Equations (Chapter 5): The practical heart of algebra. You will master techniques for solving equations with one variable, two variables, and systems of equations, with plenty of real-world applications.
5. Beyond Linear: A Preview (Chapter 6): A glimpse of what comes next—quadratic equations, polynomials, and the rich landscape of mathematics that awaits you.

1.4 How Mathematicians Think

One of the most important things you can learn from mathematics is not a formula or a technique—it is a *way of thinking*. Mathematicians approach problems with a distinctive mindset:

- Precision: Every word and symbol has a specific meaning. Vague language is the enemy.
- Skepticism: A claim is not true until it is *proved*. Even a million confirming examples are not enough—you need a logical argument that covers all cases.
- Abstraction: By stripping away irrelevant details, you can see the essential structure of a problem. The same equation might describe population growth, radioactive decay, or compound interest.
- Persistence: Hard problems rarely yield on the first attempt. Mathematicians expect to be stuck and see it as part of the process, not a sign of failure.

The Growth Mindset: Research in cognitive science has shown that mathematical ability is not a fixed talent—it is a skill that develops with practice. When you struggle with a problem, your brain is literally building new neural connections. The struggle *is* the learning.

1.5 A Brief History of Mathematical Thinking

Mathematics has a history stretching back thousands of years:

- Ancient Babylon (c. 1800 BCE): Scribes solved quadratic equations using geometric methods on clay tablets. They had tables of squares and reciprocals and could solve systems of equations.
- Ancient Greece (c. 300 BCE): Euclid wrote the *Elements*, one of the most influential textbooks ever written. He showed that all of geometry could be derived from just five axioms using pure logic. This axiomatic method is still the gold standard in mathematics.
- India (c. 500–1200 CE): Indian mathematicians invented the decimal number system and the concept of zero, which revolutionized computation. Brahmagupta gave rules for arithmetic with negative numbers and zero. Bhaskara II solved quadratic and cubic equations.
- The Islamic Golden Age (c. 800–1200 CE): Al-Khwarizmi wrote *Al-Kitab al-Mukhtasar fi Hisab al-Jabr wal-Muqabala* (“The Compendious Book on Calculation by Completion and Balancing”), which gave us the word *algebra*. His systematic methods for solving linear and quadratic equations were revolutionary.

- Renaissance Europe (c. 1500–1700 CE): Descartes unified algebra and geometry by introducing coordinates. Newton and Leibniz invented calculus. Fermat and Pascal founded probability theory.
- Modern Era: Mathematics has exploded into hundreds of subfields, from topology to cryptography to machine learning. But the foundations—logic, equations, and proof—remain the same.

Remark 1.1. Notice a pattern: every major civilization contributed to mathematics. It truly is a universal human endeavor. The tools you will learn in this book—logic, equations, systematic problem-solving—are the same tools that have been refined over millennia.

Chapter 2

Mathematical Logic

If mathematics is a language, then logic is its grammar. You can have brilliant mathematical intuition, but without logical precision, you cannot communicate your ideas or verify that they are correct. In this chapter, we build the foundations of mathematical reasoning from the ground up.

2.1 Sentences and Statements

Definition 2.1 (Statement (Proposition)). A statement (also called a proposition) is a declarative sentence that is either true or false, but not both. The truth or falsity of a statement is called its truth value.

Example 2.1 (Statements and Non-Statements).

- “ $2 + 3 = 5$ ” is a statement (true).
- “Paris is the capital of Germany” is a statement (false).
- “ $x + 1 = 5$ ” is *not* a statement as written—its truth depends on the value of x . It is called an open sentence or predicate.
- “Close the door!” is not a statement (it is a command).
- “Is it raining?” is not a statement (it is a question).
- “This sentence is false” is not a statement (it is a paradox—neither true nor false consistently).

Definition 2.2 (Prime and Composite Sentences). A prime sentence (or atomic sentence) contains no logical connectives and cannot be broken down into simpler statements. A composite sentence is built from prime sentences using logical connectives.

Example 2.2. Consider these sentences:

1. “Math is a tough subject.” — This is a *prime* sentence.
2. “English is *not* a tough subject.” — *Composite* (uses “not”).
3. “Math *and* English are both tough subjects.” — *Composite* (uses “and”).
4. “*Either* Math *or* English is tough.” — *Composite* (uses “or”).
5. “*If* Math is tough, *then* English is also tough.” — *Composite* (uses “if...then”).
6. “Math is tough *if and only if* English is tough.” — *Composite* (uses “iff”).

2.2 The Five Logical Connectives

There are five fundamental connectives in logic. They are the building blocks from which all logical statements are constructed.

The Five Logical Connectives:

Connective	Symbol	Name	Read As
not	$\neg$ or $\sim$	Negation	“not p ”
and	$\wedge$	Conjunction	“ p and q ”
or	$\vee$	Disjunction	“ p or q ”
if...then	$\Rightarrow$	Implication	“ p implies q ”
if and only if	$\Leftrightarrow$	Biconditional	“ p iff q ”

2.2.1 Negation (NOT)

The negation of a statement p is written $\neg p$ and has the opposite truth value.

Truth Table for Negation:

p	$\neg p$
T	F
F	T

Common Mistake

Negation does *not* mean “opposite.” The negation of “English is a tough subject” is “English is *not* a tough subject”—not “English is an easy subject.” Those are different statements! “Not tough” and “easy” are not logically equivalent.

Example 2.3 (Mathematical Negation). In mathematical notation, negation is often shown with a slash through the symbol:

- $x = y$ (“ x equals y ”) becomes $x \neq y$ (“ x does not equal y ”)
- $x \in A$ (“ x belongs to A ”) becomes $x \notin A$ (“ x does not belong to A ”)
- $x < 5$ becomes $x \geq 5$ (not “ $x > 5$ ”—be careful!)

Remark 2.1. The negation of “ $x < 5$ ” is “ $x \geq 5$,” not “ $x > 5$.” This is a common mistake. The negation of “less than” is “greater than *or equal to*,” because we need to negate the strict inequality completely.

2.2.2 Conjunction (AND)

The conjunction of statements p and q is written $p \wedge q$ and is true only when *both* p and q are true.

Truth Table for Conjunction:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 2.4. “I am writing *and* my sister is reading” is true only if *both* parts are true. If I am writing but my sister is sleeping, the conjunction is false.

Remark 2.2. In everyday English, words like “but,” “while,” “although,” and “yet” function similarly to “and” from a logical standpoint. “I studied hard *but* I failed the exam” has the same logical structure as “I studied hard *and* I failed the exam.” The emotional connotation differs, but the truth conditions are identical.

2.2.3 Disjunction (OR)

The disjunction of statements p and q is written $p \vee q$ and is true when *at least one* of p or q is true.

Truth Table for Disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Common Mistake

In mathematics, “or” is inclusive: “ p or q ” is true even when both are true. In everyday English, “or” is often exclusive (“Would you like tea or coffee?” usually means one, not both). In logic, if we mean exclusive or, we write it explicitly as $p \oplus q$ or $(p \vee q) \wedge \neg(p \wedge q)$.

AND ($\wedge$)			OR ($\vee$)		
P	Q	$P \wedge Q$	P	Q	$P \vee Q$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	F	F	T	T
F	F	F	F	F	F

Both must be True

At least one must be True

Key Difference: AND requires both True; OR requires at least one True

Figure 2.1: Truth tables for conjunction ($\wedge$) and disjunction ($\vee$): AND requires both inputs to be true, while OR requires at least one.

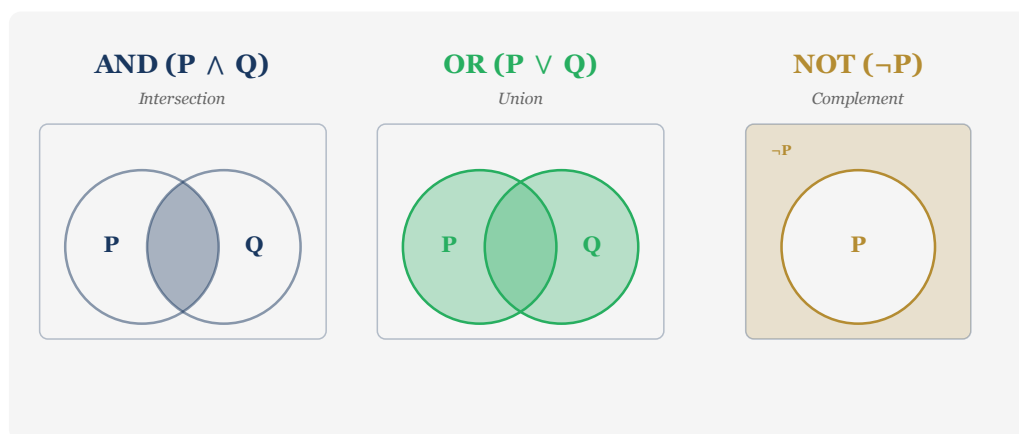


Figure 2.2: Venn diagram representation of the logical connectives: conjunction ($p \wedge q$) corresponds to the intersection, while disjunction ($p \vee q$) corresponds to the union.

Example 2.5. “It is raining or it is sunny” is true if it is raining, if it is sunny, or if (oddly enough) both are happening simultaneously. The only case where the disjunction is false is when it is neither raining nor sunny.

2.2.4 Implication (IF...THEN)

The implication $p \Rightarrow q$ reads “if p then q .” The statement p is called the antecedent (or hypothesis), and q is the consequent (or conclusion).

Truth Table for Implication:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Remark 2.3. The truth table for implication is the most counterintuitive for beginners. The key insight: an implication is only false when the hypothesis is true but the conclusion is false. When the hypothesis is false, the implication is *vacuously true*—it makes no claim, so it cannot be wrong.

Think of it as a promise: “If it rains, I will bring an umbrella.” The only way this promise is *broken* is if it rains and I do *not* bring an umbrella (p true, q false). If it does not rain, my promise is not violated regardless of whether I bring an umbrella.

There are many equivalent ways to phrase an implication:

Phrasing	Meaning
If p then q	Standard form
p implies q	Direct statement
q follows from p	Emphasizes consequence
p only if q	Tricky but equivalent
p is sufficient for q	p guarantees q
q is necessary for p	Cannot have p without q

Example 2.6. “If $5 < 6$ and $6 < 7$, then $5 < 7$.” Here the antecedent is “ $5 < 6$ and $6 < 7$ ” (which is true), and the consequent is “ $5 < 7$ ” (also true). Since both are true, the implication is true.

Converse, Inverse, and Contrapositive

Given an implication $p \Rightarrow q$, three related statements are important:

Definition 2.3 (Related Conditional Statements).

- Converse: $q \Rightarrow p$ (swap hypothesis and conclusion)
- Inverse: $\neg p \Rightarrow \neg q$ (negate both)
- Contrapositive: $\neg q \Rightarrow \neg p$ (swap and negate both)

Key Fact: An implication and its contrapositive are *logically equivalent*—they always have the same truth value. The converse and inverse are logically equivalent to each other, but *not* to the original implication.

Example 2.7. Original: “If it is raining, then the ground is wet.” ($p \Rightarrow q$)

- Converse: “If the ground is wet, then it is raining.” (Not necessarily true—someone might have used a hose.)
- Inverse: “If it is not raining, then the ground is not wet.” (Also not necessarily true.)
- Contrapositive: “If the ground is not wet, then it is not raining.” (True! This is logically equivalent to the original.)

2.2.5 Biconditional (IF AND ONLY IF)

The biconditional $p \Leftrightarrow q$ reads “ p if and only if q ” (abbreviated “ p iff q ”). It is true precisely when p and q have the same truth value.

Truth Table for Biconditional:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

A biconditional $p \Leftrightarrow q$ is equivalent to $(p \Rightarrow q) \wedge (q \Rightarrow p)$ —it asserts that the implication works in *both* directions.

Example 2.8. “A triangle is right-angled if and only if one of its angles is 90° .” This combines two claims:

- If a triangle is right-angled, then one angle is 90° . (Direct)
- If one angle is 90° , then the triangle is right-angled. (Converse)

Both are true, so the biconditional is true.

2.3 Tautologies, Contradictions, and Contingencies

Definition 2.4 (Tautology, Contradiction, Contingency).

- A tautology is a compound statement that is *always true*, regardless of the truth values of its components. Example: $p \vee \neg p$ (“either it is raining or it is not raining”).
- A contradiction is a compound statement that is *always false*. Example: $p \wedge \neg p$ (“it is raining and it is not raining”).
- A contingency is a statement that is neither a tautology nor a contradiction—its truth value depends on the components.

Example 2.9 (Verifying a Tautology). Let us verify that $p \Rightarrow (p \vee q)$ is a tautology:

p	q	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Since the last column is all T, this is indeed a tautology. This makes intuitive sense: if p is true, then certainly “ p or q ” is true.

2.4 Logical Equivalences

Two statements are logically equivalent if they have the same truth value in every possible case. We write $p \equiv q$.

Important Logical Equivalences:

1. De Morgan's Laws:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \quad (2.1)$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \quad (2.2)$$

2. Double Negation: $\neg(\neg p) \equiv p$

3. Implication: $p \Rightarrow q \equiv \neg p \vee q$

4. Contrapositive: $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

5. Biconditional: $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

6. Distributive Laws:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \quad (2.3)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \quad (2.4)$$

Example 2.10 (De Morgan's Law in Action). The negation of "I will have tea *and* cake" is:

$$\neg(\text{tea} \wedge \text{cake}) \equiv \neg\text{tea} \vee \neg\text{cake}$$

"I will *not* have tea *or* I will *not* have cake" (or both). It does *not* mean "I will have neither tea nor cake"—that would be $\neg\text{tea} \wedge \neg\text{cake}$, which is the negation of "tea *or* cake."

2.5 Quantifiers

When we want to make statements about collections of objects, we use quantifiers.

Definition 2.5 (Universal and Existential Quantifiers).

- The universal quantifier $\forall$ means "for all" or "for every." The statement $\forall x P(x)$ asserts that $P(x)$ is true for every value of x in the domain.
- The existential quantifier $\exists$ means "there exists" or "for some." The statement $\exists x P(x)$ asserts that $P(x)$ is true for at least one value of x .

Example 2.11.

- $\forall x \in \mathbb{R}, x^2 \geq 0$: "For every real number x , x^2 is non-negative." (True)
- $\exists x \in \mathbb{R}, x^2 = -1$: "There exists a real number x such that $x^2 = -1$." (False—but true if we expand to $\mathbb{C}$)
- $\forall n \in \mathbb{N}, n + 1 > n$: "Every natural number is less than its successor." (True)

Negating Quantifiers:

$$\neg(\forall x P(x)) \equiv \exists x \neg P(x) \quad (2.5)$$

$$\neg(\exists x P(x)) \equiv \forall x \neg P(x) \quad (2.6)$$

The negation of “everything has property P ” is “something does not have property P .” The negation of “something has property P ” is “nothing has property P .”

Example 2.12. The negation of “All students passed the exam” is “There exists a student who did not pass the exam”—*not* “No students passed the exam.”

2.6 Stronger and Weaker Statements

Definition 2.6 (Stronger and Weaker). Statement p is stronger than statement q if $p \Rightarrow q$ is true. Equivalently, q is weaker than p . If additionally $q \not\Rightarrow p$, then p is strictly stronger than q .

Example 2.13. “This quadrilateral is a rhombus” is strictly stronger than “This quadrilateral is a parallelogram.” Every rhombus is a parallelogram, but not every parallelogram is a rhombus.

2.7 Inductive vs. Deductive Reasoning

These are two fundamentally different ways of arriving at conclusions.

Definition 2.7 (Inductive and Deductive Reasoning).

- Inductive reasoning: Observing specific cases and generalizing to a broader conclusion. Used in science and everyday life. Conclusions are *probable* but not certain.
- Deductive reasoning: Starting from established truths (axioms, definitions, previously proved theorems) and deriving new truths through logical steps. Conclusions are *certain* if the premises are true.

Example 2.14 (The Triangle Angle Sum). Imagine measuring the angles of hundreds of triangles:

- A layperson finds the sum is 180° in most cases, ignores exceptions, and concludes: “The angles sum to 180° .” (Hasty generalization)
- A physicist attributes small deviations to measurement error and states a law: “The angles of any triangle sum to 180° .” (Inductive, with error analysis)
- A mathematician refuses to generalize from any finite number of examples. Even a million confirming cases prove nothing about the infinitely many triangles yet to be tested. Instead, they *prove* the result deductively from Euclid’s axioms.

2.8 Axioms, Postulates, and Proof

Definition 2.8 (Axiom). An axiom (or postulate) is a statement accepted as true without proof. Axioms serve as the starting points from which all theorems are derived.

Every branch of mathematics rests on its own set of axioms. Euclidean geometry, for example, is built from approximately five axioms, including “Through any two distinct points, exactly one straight line can be drawn.”

Remark 2.4. Axioms are not arbitrary. They are chosen because they are self-evident or because they lead to useful, consistent mathematics. But they cannot be proved—that is precisely what makes them axioms.

2.8.1 Common Proof Techniques

Here is a preview of the main methods mathematicians use to prove theorems:

Definition 2.9 (Proof Techniques).

1. Direct Proof: Assume p is true. Using logical steps, derive that q is true. This proves $p \Rightarrow q$.
2. Proof by Contrapositive: Instead of proving $p \Rightarrow q$, prove the logically equivalent $\neg q \Rightarrow \neg p$.
3. Proof by Contradiction: Assume $\neg p$ (or assume the negation of what you want to prove). Derive a contradiction. Conclude that p must be true.
4. Mathematical Induction: To prove a statement $P(n)$ for all natural numbers $n \geq n_0$:
 - Base case: Verify $P(n_0)$ is true.
 - Inductive step: Assume $P(k)$ is true for some $k \geq n_0$ (the *inductive hypothesis*). Prove that $P(k + 1)$ is true.

Example 2.15 (Direct Proof). Claim: If n is even, then n^2 is even.

Proof: Assume n is even. Then $n = 2k$ for some integer k . Therefore $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, n^2 is even. $\square$

Example 2.16 (Proof by Contradiction). Claim: $\sqrt{2}$ is irrational.

Proof: Assume, for contradiction, that $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{a}{b}$ where a, b are integers with no common factors. Squaring: $2 = \frac{a^2}{b^2}$, so $a^2 = 2b^2$. This means a^2 is even, so a is even (by the previous example). Write $a = 2c$. Then $(2c)^2 = 2b^2$, so $4c^2 = 2b^2$, giving $b^2 = 2c^2$. Thus b^2 is even, so b is even. But then both a and b are even, contradicting our assumption that they have no common factors. Therefore $\sqrt{2}$ is irrational. $\square$

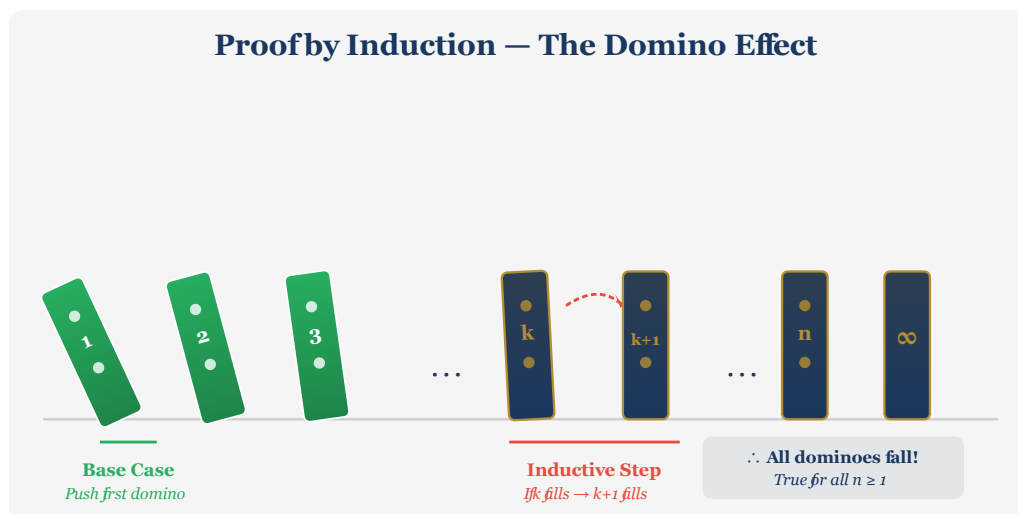


Figure 2.3: Mathematical induction visualised as falling dominoes: if the first domino falls (base case) and each domino knocks over the next (inductive step), then all dominoes fall.

Example 2.17 (Mathematical Induction). Claim: For all $n \geq 1$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Proof:

- Base case ($n = 1$): $1 = \frac{1 \cdot 2}{2} = 1$. $\square$
- Inductive step: Assume $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ for some $k \geq 1$. Then:

$$1 + 2 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1) \quad (2.7)$$

$$= \frac{k(k + 1) + 2(k + 1)}{2} \quad (2.8)$$

$$= \frac{(k + 1)(k + 2)}{2} \quad (2.9)$$

This is the formula with $n = k + 1$. $\square$

By induction, the formula holds for all $n \geq 1$. $\square$

2.9 Arguments, Premises, and Conclusions

Definition 2.10 (Argument). An argument consists of a set of premises (hypotheses) and a conclusion. An argument is valid if the conclusion follows logically from the premises. Symbolically, if the premises are $p_1, p_2, \dots, p_n$ and the conclusion is q , the argument is the implication:

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \Rightarrow q$$

Example 2.18 (The Socrates Argument). • Premise p_1 : Every man is mortal.

- Premise p_2 : Socrates is a man.
- Conclusion q : Socrates is mortal.

This is a valid argument: $(p_1 \wedge p_2) \Rightarrow q$ is a tautology.

Exercise 2.1. Determine whether the following argument is valid:

- Premise: If it is raining, then the streets are wet.
- Premise: The streets are wet.
- Conclusion: It is raining.

(Hint: This is a common logical fallacy called “affirming the consequent.”)

Exercise 2.2. Construct truth tables for each of the following:

1. $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
2. $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$
3. $(p \vee q) \wedge \neg p \Rightarrow q$

Which of these are tautologies?

Chapter 3

Numbers and Measurement: Significant Figures

In the real world, measurements are never perfectly precise. When a scientist says a sample weighs “12.34 grams,” they are communicating not just a value but a *level of precision*. The concept of significant figures makes this communication possible. In this chapter, we learn the rules, the reasoning, and the applications.

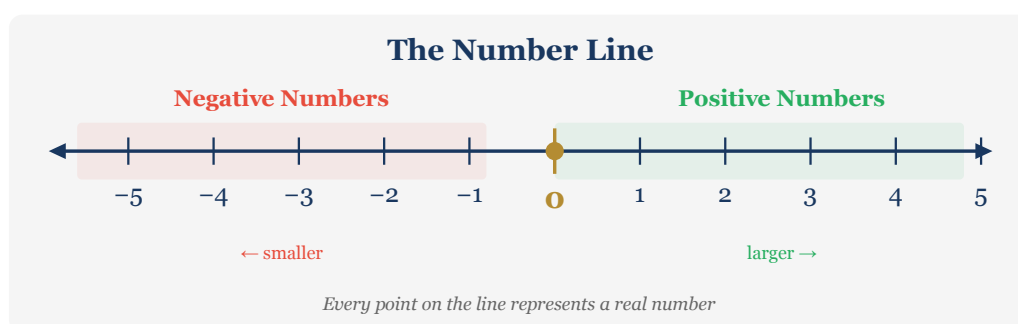


Figure 3.1: The real number line: integers, fractions, and irrational numbers all find their place on this continuous one-dimensional continuum.

3.1 What Are Significant Figures?

Definition 3.1 (Significant Figures). Significant figures (also called significant digits or sig figs) are the digits in a number that carry meaningful information about the precision of a measurement. They include all non-zero digits, any zeros between significant digits, and trailing zeros that indicate measured precision.

Example 3.1 (Why Significant Figures Matter). Suppose you measure the length of a table with a ruler marked in centimeters and get 152 cm. Your friend measures the same table with a laser device and gets 152.37 cm. These two measurements have different levels of precision:

- 152 cm has 3 significant figures (precision to the nearest cm).
- 152.37 cm has 5 significant figures (precision to the nearest hundredth of a cm).

If you average them and report “152.185 cm” (6 significant figures), you are falsely claiming a precision that your ruler measurement does not support. Significant figures help you avoid this dishonesty.

3.2 The Significance of Significant Figures

Why should you care about counting digits? Because in science and engineering, every measurement has uncertainty. When you write a number, the number of significant figures tells the reader how much of that number is actually *known*.

Consider a distance of 1200 m. Does this mean:

- Exactly 1200 m? (4 significant figures)
- Approximately 1200 m, measured to the nearest hundred? (2 significant figures)
- Measured to the nearest ten? (3 significant figures)

Without the concept of significant figures (or scientific notation), there is no way to tell. This ambiguity can cause serious errors in engineering and scientific calculations.

3.3 Rules for Counting Significant Digits

The Four Rules of Significant Figures:

1. All non-zero digits are significant.
Example: 500079 has 6 significant digits.
2. Leading zeros are never significant. They merely indicate the position of the decimal point.
Example: 0.000012 has 2 significant digits (only 1 and 2).
3. Captive zeros (between non-zero digits) are always significant.
Example: 10.04 has 4 significant digits.
4. Trailing zeros are significant only if there is a decimal point.
Example: 0.0001200 has 4 significant digits (1, 2, and two trailing zeros).
Example: 120000 without a decimal point has 2 significant figures (ambiguous—could be written as 1.2×10^5).

Example 3.2 (Counting Significant Figures).

Number	Sig Figs	Explanation
4053	4	All non-zero + captive zero
0.0072	2	Leading zeros not significant
8.00	3	Trailing zeros after decimal are significant
1000	1	Trailing zeros, no decimal (ambiguous, conservative)
1000.	4	The decimal point indicates trailing zeros are significant
3.040	4	Captive zero + trailing zero after decimal
6.022×10^{23}	4	Only the coefficient matters in scientific notation

Pro Tip

When in doubt about trailing zeros, use scientific notation. It removes all ambiguity:

- 1.2×10^5 clearly has 2 significant figures.
- 1.20×10^5 clearly has 3 significant figures.
- 1.200×10^5 clearly has 4 significant figures.

3.4 Scientific Notation

Definition 3.2 (Scientific Notation). A number is in scientific notation when it is written as $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer.

Example 3.3 (Converting to Scientific Notation).

- $45,000 = 4.5 \times 10^4$ (2 sig figs, or 4.50×10^4 for 3 sig figs)
- $0.00327 = 3.27 \times 10^{-3}$ (3 sig figs)
- $6,022,000,000,000,000,000,000 = 6.022 \times 10^{23}$ (Avogadro's number, 4 sig figs)

3.5 Operations with Significant Figures

When you perform calculations with measured values, you must propagate the precision correctly.

3.5.1 Addition and Subtraction

Rule for Addition/Subtraction: The result should have the same number of decimal places as the measurement with the fewest decimal places.

Example 3.4. Calculate $12.11 + 0.3 + 1.456$:

$$12.11 \quad (2 \text{ decimal places}) \quad (3.1)$$

$$0.3 \quad (1 \text{ decimal place — fewest}) \quad (3.2)$$

$$+ 1.456 \quad (3 \text{ decimal places}) \quad (3.3)$$

$$13.866 \quad \text{Round to 1 decimal place: } \boxed{13.9} \quad (3.4)$$

3.5.2 Multiplication and Division

Rule for Multiplication/Division: The result should have the same number of significant figures as the measurement with the fewest significant figures.

Example 3.5. Calculate 4.56×1.4 :

- 4.56 has 3 significant figures.
- 1.4 has 2 significant figures (fewest).
- $4.56 \times 1.4 = 6.384$
- Round to 2 significant figures: $\boxed{6.4}$

Example 3.6. Calculate $\frac{0.0250}{3.0}$:

- 0.0250 has 3 significant figures.
- 3.0 has 2 significant figures (fewest).
- $\frac{0.0250}{3.0} = 0.008\bar{3}$
- Round to 2 significant figures: $\boxed{0.0083}$

3.6 Rounding Rules

Rounding Significant Figures:

1. Identify the last significant digit you want to keep.
2. Look at the digit immediately to its right:
 - If it is less than 5, leave the last significant digit unchanged (“round down”).
 - If it is greater than 5, increase the last significant digit by 1 (“round up”).
 - If it is exactly 5, round to the nearest *even* digit (“banker’s rounding”)—this avoids systematic bias over many calculations.
3. Drop all digits to the right.

Example 3.7 (Rounding Practice).

Number	Round to	Result
3.456	3 sig figs	3.46
3.454	3 sig figs	3.45
3.455	3 sig figs	3.46 (round to even)
3.445	3 sig figs	3.44 (round to even)
0.008274	2 sig figs	0.0083
1285	3 sig figs	1280 or 1.28×10^3

3.7 Precision vs. Accuracy

These two terms are often confused but mean different things:

Definition 3.3 (Precision and Accuracy).

- Precision measures how close repeated measurements are to *each other*. High precision means low scatter.
- Accuracy measures how close a measurement is to the *true value*. High accuracy means low systematic error.

Example 3.8 (Archery Analogy). Imagine shooting arrows at a target:

- High accuracy, high precision: All arrows clustered near the bullseye.
- High accuracy, low precision: Arrows scattered but centered around the bullseye.
- Low accuracy, high precision: All arrows clustered together, but far from the bullseye.
- Low accuracy, low precision: Arrows scattered everywhere.

3.8 Applications in Science and Engineering

Example 3.9 (Chemistry: Molar Mass Calculation). The atomic mass of carbon is 12.011 g/mol (5 sig figs) and hydrogen is 1.008 g/mol (4 sig figs). For methane (CH₄):

$$\text{Molar mass} = 12.011 + 4 \times 1.008 \quad (3.5)$$

$$= 12.011 + 4.032 \quad (3.6)$$

$$= 16.043 \text{ g/mol} \quad (3.7)$$

We report 5 significant figures because both inputs have at least 4–5 sig figs (addition rule: match decimal places).

Example 3.10 (Physics: Speed Calculation). A car travels 150. m (3 sig figs) in 6.2 s (2 sig figs).

$$\text{Speed} = \frac{150.}{6.2} = 24.19 \dots \approx \boxed{24 \text{ m/s}}$$

We round to 2 significant figures because 6.2 has the fewest.

Example 3.11 (Engineering: Area Calculation). A rectangular plot measures 45.3 m by 12 m.

$$\text{Area} = 45.3 \times 12 = 543.6 \approx \boxed{540 \text{ m}^2}$$

We round to 2 significant figures (the precision of 12 m).

3.9 Practice Problems

Exercise 3.1. How many significant figures are in each of the following?

1. 0.00340
2. 5.060
3. 200
4. 200.
5. 1.00×10^{-5}
6. 90100

Exercise 3.2. Perform each calculation and express the result with the correct number of significant figures:

1. $12.5 + 1.37 + 0.9$
2. 8.4×3.25
3. $\frac{45.0}{2.1}$
4. $100.0 - 7.56$

Exercise 3.3. A student measures the dimensions of a box as 10.2 cm $\times$ 5.0 cm $\times$ 3.15 cm. Calculate the volume with the appropriate number of significant figures.

Chapter 4

Equations: A Basic Introduction

Math teachers love to say “let x equal the unknown.” But nobody explains why that simple trick is so powerful. Here is the thing: equations are the reason we can solve problems that would take hours of guesswork in seconds.

4.1 Why Equations Matter

Applied mathematics is not abstract theory—it is the toolkit you use to solve real problems. Business decisions, engineering calculations, financial planning—they all boil down to equations.

Before we dive into the theory, let us see equations in action.

Example 4.1 (George’s Money). Problem: George had some money. He gave \$14 to Matthew. Now he has \$27. How much did George start with?

You probably already know the answer: \$41. But the power of equations is that they provide a *method* that scales to harder problems.

Step 1: Translate words into symbols.

Let x = the amount George started with.

What Happened	Mathematical Expression
George started with	x dollars
He gave Matthew	14 dollars
He now has	$x - 14$ dollars

The problem says he has \$27 left, so:

$$x - 14 = 27$$

Step 2: Solve for x .

Add 14 to both sides:

$$x - 14 + 14 = 27 + 14$$

$$x = 41$$

George had \$41. ✓

4.2 What Exactly Is an Equation?

Definition 4.1 (Equation). An equation is a mathematical statement asserting that two expressions are equal. It contains:

- Variables: Unknown quantities, usually represented by x, y, z, θ, ω .
- Constants: Known values, often represented by a, b, c, d .
- An equals sign: The bridge that makes it an equation, not merely an expression.

Solving an equation means finding the value(s) of the variable that make the statement true.

Common Mistake

An expression is not an equation. The expression $3x + 5$ has no equals sign—it is a mathematical phrase, not a statement. You can simplify expressions, but you *solve* equations.

4.3 The Golden Rule of Equations

The Golden Rule: Whatever you do to one side of an equation, you must do to the other side. This keeps the equation balanced.

The four basic operations and their inverses:

Operation	What It Does	When Moving Across =
Addition (+)	Increases a value	Becomes subtraction
Subtraction (−)	Decreases a value	Becomes addition
Multiplication (×)	Scales a value	Becomes division
Division (÷)	Splits a value	Becomes multiplication

Pro Tip

When you “move a term across the equals sign and flip its sign,” you are really performing the same operation on both sides and then simplifying. The shortcut just skips the intermediate step.

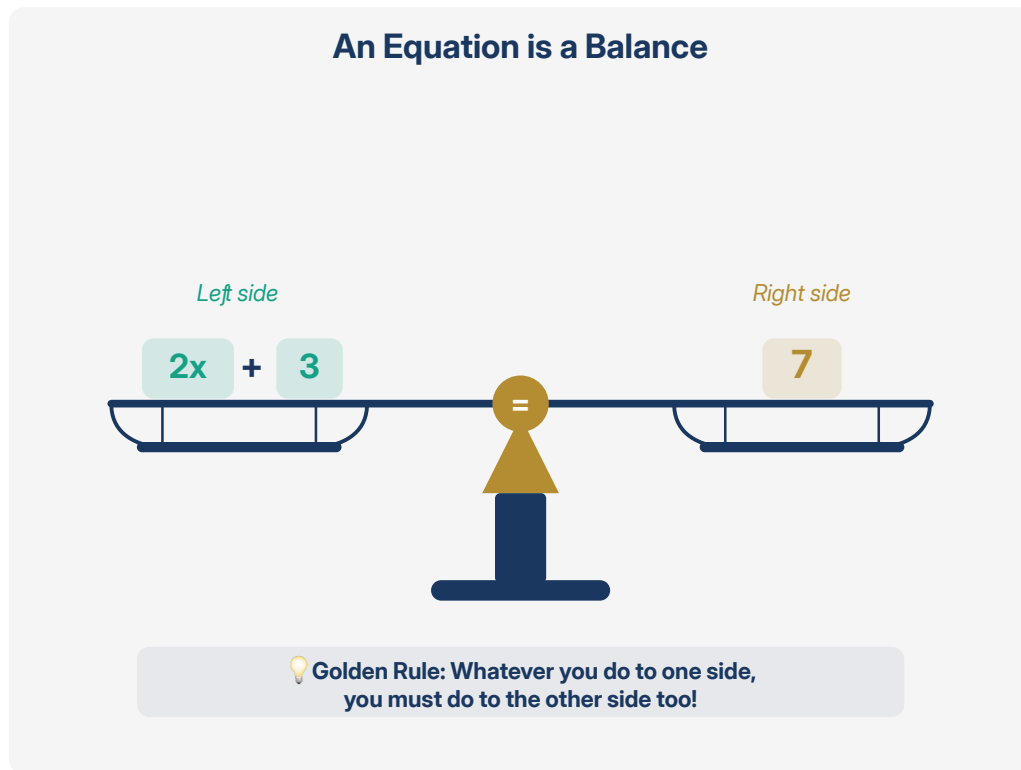


Figure 4.1: An equation as a balance scale: whatever operation is performed on one side must also be performed on the other to keep the equation balanced.

4.4 More Worked Examples

Example 4.2 (Mary's Sheep). Problem: Mary had 7 sheep. Her uncle gifted her some more. She now has 18 sheep. How many did her uncle give her?

Let x = number of sheep from her uncle.

$$7 + x = 18$$

$$x = 18 - 7 = 11$$

Wait—let's recheck. $7 + 11 = 18$. ✓

Mary's uncle gifted her 11 sheep.

Example 4.3 (The Cricket Ball Problem). Problem: Monty has some cricket balls. Graham has twice as many as Monty. Adam has 6 balls. Together, they have 27 balls. How many does each person have?

Step 1: Define variables.

- Monty has x balls.
- Graham has $2x$ balls (twice Monty's).
- Adam has 6 balls.

Step 2: Write the equation.

$$x + 2x + 6 = 27$$

Step 3: Simplify and solve.

$$3x + 6 = 27$$

$$3x = 27 - 6 = 21$$

$$x = \frac{21}{3} = 7$$

Answer: Monty has 7, Graham has 14, Adam has 6.

Check: $7 + 14 + 6 = 27$. ✓

Example 4.4 (Age Problem). Problem: Sarah is 5 years older than her brother Tom. The sum of their ages is 31. How old is each?

Let Tom's age = t .

Then Sarah's age = $t + 5$.

$$t + (t + 5) = 31$$

$$2t + 5 = 31$$

$$2t = 26$$

$$t = 13$$

Tom is 13, Sarah is 18. Check: $13 + 18 = 31$. ✓

4.5 Types of Equations: A Complete Classification

Algebraic equations are classified in two main ways: by the number of variables and by the degree (highest power of the variable).

4.5.1 Classification by Number of Variables

Type	Variables	Examples
Univariable	1	$3x + 2 = 11$, $x^2 - 4 = 0$
Bivariable	2	$x + y = 5$, $x^2 + y^2 = 25$
Trivariable	3	$x + y + z = 10$
n -variable	n	Equations with n different unknowns

4.5.2 Classification by Degree

Definition 4.2 (Degree of an Equation). The degree of a polynomial equation is the highest power of the variable that appears. The degree determines how many solutions the equation can have and what methods are needed to solve it.

Name	Degree	General Form	Max Solutions
Linear	1	$ax + b = 0$	1
Quadratic	2	$ax^2 + bx + c = 0$	2
Cubic	3	$ax^3 + bx^2 + cx + d = 0$	3
Quartic	4	$ax^4 + \dots + e = 0$	4
n th degree	n	Highest power is x^n	n

In all cases, the leading coefficient $a \neq 0$; otherwise the equation drops to a lower degree.

4.6 Identities vs. Conditional Equations

Definition 4.3 (Identity and Conditional Equation).

- An identity is an equation that is true for *all* values of the variable. Example: $2(x+1) = 2x+2$. Expand the left side and you get the right side, no matter what x is.
- A conditional equation is true only for specific values. Example: $2x = 10$ is true only when $x = 5$.
- A contradiction is an equation that is true for *no* values. Example: $x+1 = x+2$ simplifies to $1 = 2$, which is never true.

Example 4.5 (Identifying Equation Types).

1. $3(x-2) = 3x-6$: Expand the left: $3x-6 = 3x-6$. True for all x . Identity.
2. $5x+3 = 18$: Only true when $x = 3$. Conditional.
3. $x+5 = x-1$: Simplifies to $5 = -1$. Never true. Contradiction.

4.7 Solution Sets and Equivalent Equations

Definition 4.4 (Solution Set). The solution set of an equation is the set of all values that make the equation true. For $2x = 10$, the solution set is $\{5\}$. For $x^2 = 4$, the solution set is $\{-2, 2\}$. For an identity, the solution set is the entire domain.

Definition 4.5 (Equivalent Equations). Two equations are equivalent if they have the same solution set. For example, $2x = 10$ and $x = 5$ are equivalent.

Remark 4.1. When we “solve” an equation, we are really transforming it into simpler equivalent equations until we reach one of the form $x = \text{value}$.

4.8 The Power of Variables: Why Letters?

You might wonder: why use letters like x and y instead of just question marks?

Letters let you *manipulate* unknowns using the same rules as numbers. You can add x to both sides, multiply 3 by x to get $3x$, or substitute a value for x . Try doing that with a question mark! The convention of using x, y, z for variables and a, b, c for constants comes from René Descartes in the 1600s.

Exercise 4.1. For each equation, determine whether it is an identity, a conditional equation, or a contradiction:

1. $4(x+3) = 4x+12$
2. $2x+1 = 2x+7$
3. $3x-9 = 0$
4. $(x+1)^2 = x^2+2x+1$
5. $\frac{x}{x} = 1$ (careful—what about $x = 0$?)

Exercise 4.2. Translate each word problem into an equation (do not solve):

1. A number increased by 7 equals 22.

2. Three times a number decreased by 4 equals 17.
3. The perimeter of a rectangle with width w and length $2w + 3$ is 46 cm.
4. A store sells shirts for \$25 each and pants for \$40 each. You buy some of each and spend \$230 total.

Chapter 5

Solving Linear Equations

Welcome to the practical heart of algebra! Linear equations are the most fundamental type of equation, and mastering them is the key to everything that follows. In this chapter, we progress from the basics through multi-step equations to systems of equations with two variables.

5.1 What Is a Linear Equation?

Definition 5.1 (Linear Equation). A linear equation is an equation in which every variable appears only to the first power (no squares, cubes, or higher powers) and no variables are multiplied together. When graphed, a linear equation in two variables produces a straight line.

The simplest form is $ax + b = c$, where a , b , and c are known constants and x is the unknown.

5.2 Forms of Linear Equations

Linear equations appear in several standard forms:

5.2.1 Standard Form

$$Ax + By = C$$

where A , B , C are constants. Example: $3x + 4y = 12$.

5.2.2 Slope-Intercept Form

$$y = mx + b$$

where m is the slope (steepness) and b is the y -intercept (where the line crosses the y -axis).

Example 5.1. In $y = 2x + 3$, the slope is $m = 2$ (for every 1 unit increase in x , y increases by 2) and the y -intercept is $b = 3$ (the line crosses the y -axis at $(0, 3)$).

5.2.3 Point-Slope Form

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is a known point on the line and m is the slope.

Example 5.2. A line through the point $(3, 4)$ with slope 2:

$$y - 4 = 2(x - 3)$$

Expanding: $y = 2x - 6 + 4 = 2x - 2$.

5.2.4 Intercept Form

$$\frac{x}{a} + \frac{y}{b} = 1$$

where a is the x -intercept and b is the y -intercept.

5.2.5 Special Cases: Horizontal and Vertical Lines

- $y = b$ is a horizontal line (slope = 0).
- $x = a$ is a vertical line (undefined slope).

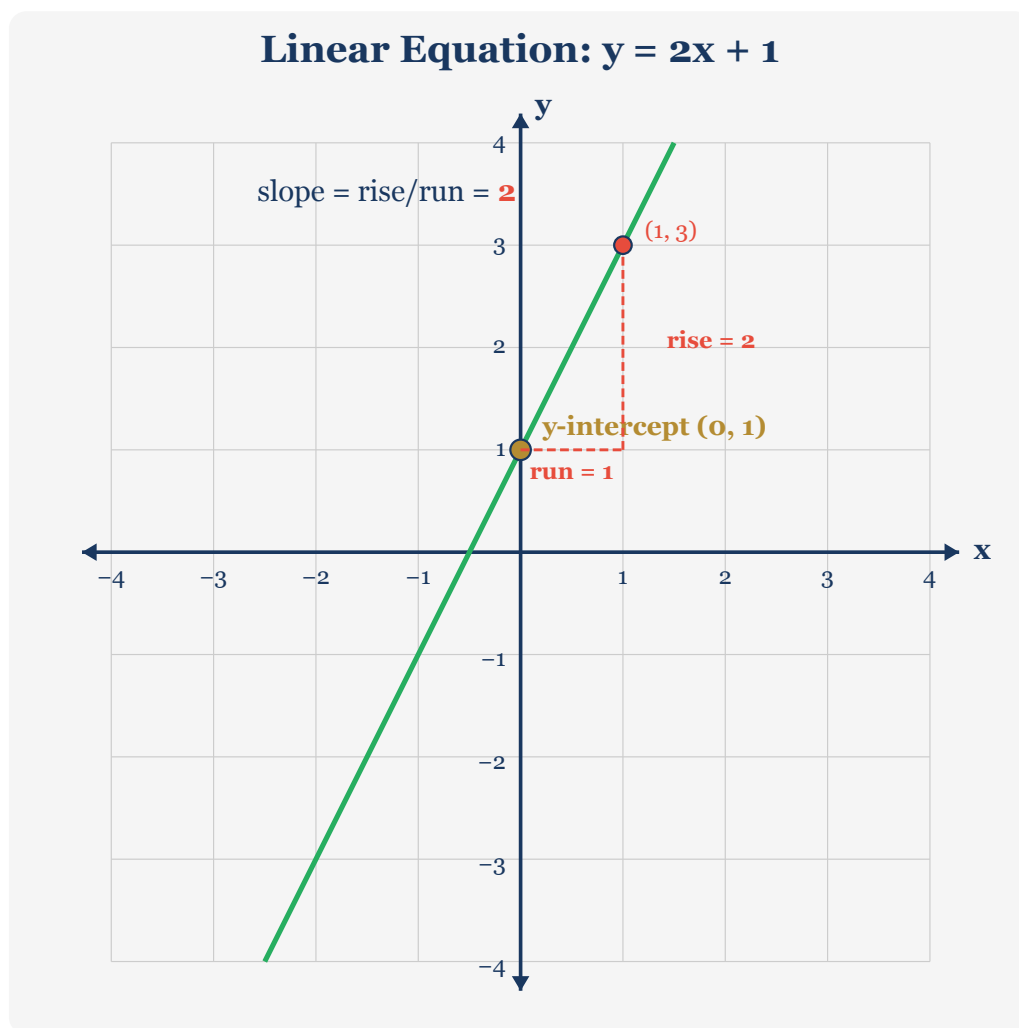


Figure 5.1: Linear equations graphed on the coordinate plane: each equation $y = mx + b$ produces a straight line with slope m and y -intercept b .

5.3 Solving One-Variable Linear Equations

5.3.1 Basic Steps

Strategy for Solving $ax + b = c$:

1. Simplify each side (distribute, combine like terms).
2. Move all variable terms to one side and constants to the other.
3. Divide (or multiply) to isolate x .
4. Check your answer by substituting back.

Example 5.3 (One-Step Equation). Solve $3x = 18$.

$$x = \frac{18}{3} = 6$$

Check: $3(6) = 18$. ✓

Example 5.4 (Two-Step Equation). Solve $3x + 4 = 10$.

$$3x + 4 = 10 \quad (5.1)$$

$$3x = 10 - 4 = 6 \quad (5.2)$$

$$x = \frac{6}{3} = 2 \quad (5.3)$$

Check: $3(2) + 4 = 6 + 4 = 10$. ✓

Example 5.5 (Equation with Distribution). Solve $2(x + 3) = 14$.

$$2x + 6 = 14 \quad (5.4)$$

$$2x = 14 - 6 = 8 \quad (5.5)$$

$$x = \frac{8}{2} = 4 \quad (5.6)$$

Check: $2(4 + 3) = 2(7) = 14$. ✓

5.3.2 Multi-Step Equations

Example 5.6 (Multi-Step). Solve $3(2x - 4) + 6 = 24$.

$$6x - 12 + 6 = 24 \quad (\text{distribute}) \quad (5.7)$$

$$6x - 6 = 24 \quad (\text{combine like terms}) \quad (5.8)$$

$$6x = 30 \quad (\text{add 6 to both sides}) \quad (5.9)$$

$$x = 5 \quad (\text{divide by 6}) \quad (5.10)$$

Check: $3(2 \cdot 5 - 4) + 6 = 3(10 - 4) + 6 = 3(6) + 6 = 18 + 6 = 24$. ✓

Example 5.7 (Variables on Both Sides). Solve $5x + 3 = 2x + 18$.

$$5x - 2x + 3 = 18 \quad (\text{subtract } 2x \text{ from both sides}) \quad (5.11)$$

$$3x + 3 = 18 \quad (5.12)$$

$$3x = 15 \quad (5.13)$$

$$x = 5 \quad (5.14)$$

Check: $5(5) + 3 = 28$ and $2(5) + 18 = 28$. ✓

Example 5.8 (Equation with Fractions). Solve $\frac{2}{3}x - 5 = 1$.

$$\frac{2}{3}x = 1 + 5 = 6 \quad (5.15)$$

$$x = 6 \times \frac{3}{2} = 9 \quad (5.16)$$

Check: $\frac{2}{3}(9) - 5 = 6 - 5 = 1$. ✓

Example 5.9 (Equation with Negative Coefficients). Solve $-5x + 9 = -16$.

$$-5x = -16 - 9 = -25 \quad (5.17)$$

$$x = \frac{-25}{-5} = 5 \quad (5.18)$$

Check: $-5(5) + 9 = -25 + 9 = -16$. ✓

5.4 Properties of Equality

These are the fundamental rules that justify each step of equation solving:

Properties of Equality:

1. Addition Property: If $a = b$, then $a + c = b + c$.
2. Subtraction Property: If $a = b$, then $a - c = b - c$.
3. Multiplication Property: If $a = b$, then $ac = bc$.
4. Division Property: If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.
5. Transitive Property: If $a = b$ and $b = c$, then $a = c$.
6. Symmetric Property: If $a = b$, then $b = a$.
7. Substitution Property: If $a = b$, then b can replace a in any expression.

Additional algebraic properties you will use constantly:

- Distributive Property: $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$.
- Combining Like Terms: $ax + bx = (a + b)x$. For instance, $3x + 4x = 7x$.
- Commutative Property: $a + b = b + a$ and $ab = ba$.
- Associative Property: $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
- Identity Elements: $a + 0 = a$ and $a \cdot 1 = a$.
- Inverse Elements: $a + (-a) = 0$ and $a \cdot \frac{1}{a} = 1$ (for $a \neq 0$).

5.5 Real-World Applications

Example 5.10 (Budgeting). You have a monthly budget of \$500 for expenses. Each outing costs \$60 and you have \$140 in other fixed expenses. How many outings can you afford?

$$60n + 140 = 500$$

$$60n = 360$$

$$n = 6$$

You can afford 6 outings per month.

Example 5.11 (Physics: Distance-Speed-Time). A car travels at a constant speed of 65 km/h. How long does it take to travel 260 km?

$$65t = 260$$

$$t = \frac{260}{65} = 4 \text{ hours}$$

Example 5.12 (Cooking: Scaling a Recipe). A recipe for 4 servings requires $\frac{2}{3}$ cup of sugar. How much sugar for 10 servings?

Let s = sugar needed.

$$\frac{s}{10} = \frac{2/3}{4} = \frac{1}{6}$$

$$s = \frac{10}{6} = \frac{5}{3} \approx 1.67 \text{ cups}$$

5.6 Systems of Linear Equations

When you have two unknowns, you need two equations. A system of linear equations is a set of two or more linear equations with the same variables.

5.6.1 Graphical Interpretation

Each linear equation in two variables represents a line in the plane. The solution to a system is the point(s) where the lines intersect:

- One solution: The lines intersect at exactly one point (the system is *consistent and independent*).
- No solution: The lines are parallel and never intersect (the system is *inconsistent*).
- Infinitely many solutions: The lines are identical, overlapping completely (the system is *dependent*).

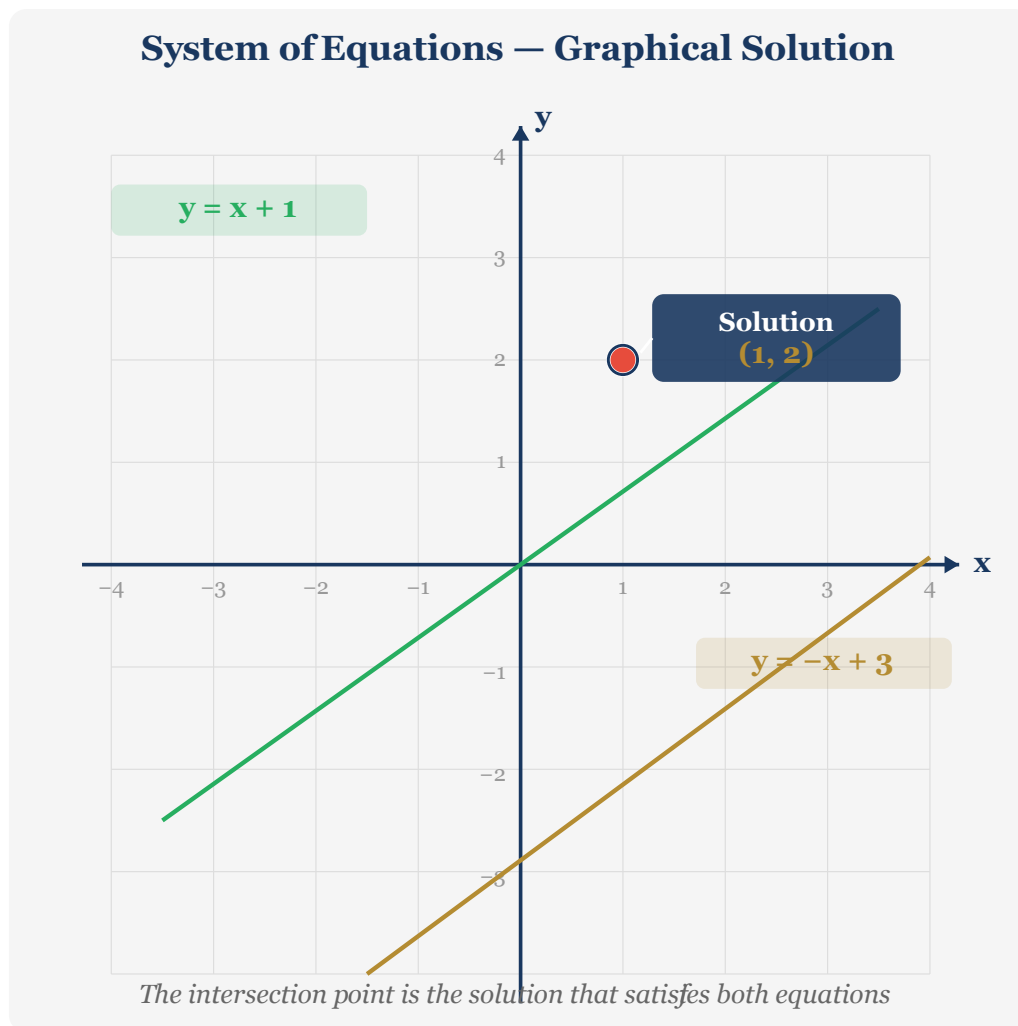


Figure 5.2: Graphical interpretation of systems of linear equations: one solution (intersecting lines), no solution (parallel lines), or infinitely many solutions (coincident lines).

5.6.2 Substitution Method

Substitution Method:

1. Solve one equation for one variable.
2. Substitute that expression into the other equation.
3. Solve the resulting single-variable equation.
4. Substitute back to find the other variable.

Example 5.13 (Substitution). Solve the system:

$$x + y = 5 \quad (5.19)$$

$$2x - y = 4 \quad (5.20)$$

Step 1: From (5.19): $y = 5 - x$.

Step 2: Substitute into (5.20):

$$2x - (5 - x) = 4$$

$$2x - 5 + x = 4$$

$$3x = 9$$

$$x = 3$$

Step 3: $y = 5 - 3 = 2$.

Solution: $(x, y) = (3, 2)$.

Check: $3 + 2 = 5$ ✓ and $2(3) - 2 = 4$ ✓.

5.6.3 Elimination Method

Elimination Method:

1. Multiply one or both equations so that one variable has equal (or opposite) coefficients.
2. Add or subtract the equations to eliminate that variable.
3. Solve for the remaining variable.
4. Substitute back to find the eliminated variable.

Example 5.14 (Elimination). Solve the system:

$$x + y = 10 \quad (5.21)$$

$$2x - y = 3 \quad (5.22)$$

Step 1: The y coefficients are $+1$ and -1 . Add the equations:

$$(x + y) + (2x - y) = 10 + 3$$

$$3x = 13$$

$$x = \frac{13}{3}$$

Step 2: $y = 10 - \frac{13}{3} = \frac{30-13}{3} = \frac{17}{3}$.

Solution: $(x, y) = \left(\frac{13}{3}, \frac{17}{3}\right)$.

Check: $\frac{13}{3} + \frac{17}{3} = \frac{30}{3} = 10$ ✓ and $\frac{26}{3} - \frac{17}{3} = \frac{9}{3} = 3$ ✓.

Example 5.15 (Word Problem with System). Problem: A school sells student tickets for \$5 and adult tickets for \$8. They sell 20 tickets total and collect \$130. How many of each?

Let s = student tickets, a = adult tickets.

$$s + a = 20 \quad (5.23)$$

$$5s + 8a = 130 \quad (5.24)$$

From (5.23): $s = 20 - a$.

Substitute into (5.24):

$$5(20 - a) + 8a = 130$$

$$100 - 5a + 8a = 130$$

$$3a = 30$$

$$a = 10$$

So $s = 20 - 10 = 10$.

Answer: 10 student tickets and 10 adult tickets.

Check: $10 + 10 = 20$ ✓ and $5(10) + 8(10) = 50 + 80 = 130$ ✓.

5.6.4 Preview: Matrix Methods

For systems with many variables, the methods above become tedious. Mathematicians have developed matrix methods that systematically organize and solve large systems. The key idea is to represent the system as a matrix equation:

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

Using row reduction (also called Gaussian elimination), you can solve any system of linear equations, no matter how many variables. This technique is fundamental in linear algebra, a subject you will encounter in college-level mathematics.

5.7 Linear Inequalities

A linear inequality looks like a linear equation, but with $<$, $>$, $\leq$, or $\geq$ instead of $=$.

Key Difference: When you multiply or divide both sides of an inequality by a negative number, you must flip the inequality sign.

Example 5.16 (Solving an Inequality). Solve $3x + 2 > 8$.

$$3x > 8 - 2 = 6 \quad (5.25)$$

$$x > 2 \quad (5.26)$$

The solution set is all real numbers greater than 2, written $(2, \infty)$.

Example 5.17 (Inequality with Sign Flip). Solve $-4x + 3 \leq 11$.

$$-4x \leq 11 - 3 = 8 \quad (5.27)$$

$$x \geq \frac{8}{-4} = -2 \quad (\text{flip the sign!}) \quad (5.28)$$

The solution set is $[-2, \infty)$.

5.8 Practice Problems

Exercise 5.1. Solve for x :

1. $7x + 3 = 31$

2. $4(2x - 3) = 28$

3. $2x + 7 = 3x - 2$

4. $3x + 5 + 7x - 2 = 45$

5. $\frac{x+3}{4} = \frac{2x-1}{3}$

Exercise 5.2. Solve each system of equations:

1.
$$\begin{cases} x + y = 8 \\ x - y = 2 \end{cases}$$

$$2. \begin{cases} 3x + 2y = 12 \\ x - y = 1 \end{cases}$$

$$3. \begin{cases} 2x + 3y = 7 \\ 4x + 6y = 14 \end{cases} \quad (\text{What happens here?})$$

Exercise 5.3 (Word Problems). 1. A rectangle's perimeter is 48 cm. Its length is 3 cm more than twice its width. Find the dimensions.

2. Two trains leave a station at the same time, traveling in opposite directions. One travels at 60 km/h and the other at 90 km/h. After how many hours will they be 450 km apart?

3. A phone plan charges a \$25 monthly fee plus \$0.10 per text message. Another plan charges \$15 monthly plus \$0.15 per text. For how many texts per month are the plans equally expensive?

Exercise 5.4. Given a point $(4, -2)$ and a slope of 3, write the equation of the line in:

1. Point-slope form
2. Slope-intercept form
3. Standard form

Chapter 6

Beyond Linear: A Preview

You have now mastered the foundations: logic, measurement, and linear equations. But mathematics does not stop at straight lines. In this chapter, we give you a taste of what lies ahead—the beautiful and powerful world of nonlinear equations.

6.1 Quadratic Equations

Definition 6.1 (Quadratic Equation). A quadratic equation has the general form:

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

The word “quadratic” comes from the Latin *quadratus*, meaning “square,” because the variable is squared.

Example 6.1. $x^2 + 5x + 6 = 0$ is a quadratic equation with $a = 1$, $b = 5$, $c = 6$.

6.1.1 Solving by Factoring

If you can express the quadratic as a product of two linear factors, you can solve it using the zero product property: if $AB = 0$, then $A = 0$ or $B = 0$.

Example 6.2. Solve $x^2 + 5x + 6 = 0$.

We look for two numbers that multiply to 6 and add to 5. Those numbers are 2 and 3:

$$x^2 + 5x + 6 = (x + 2)(x + 3) = 0$$

So $x + 2 = 0$ or $x + 3 = 0$, giving $x = -2$ or $x = -3$.

Check: $(-2)^2 + 5(-2) + 6 = 4 - 10 + 6 = 0$ ✓

$(-3)^2 + 5(-3) + 6 = 9 - 15 + 6 = 0$ ✓

6.1.2 The Quadratic Formula

Not every quadratic factors neatly. For the general case, we have:

The Quadratic Formula: The solutions of $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $\Delta = b^2 - 4ac$ is called the discriminant:

- $\Delta > 0$: Two distinct real solutions.
- $\Delta = 0$: One repeated real solution.
- $\Delta < 0$: No real solutions (two complex solutions).

Example 6.3. Solve $2x^2 - 3x - 2 = 0$.

Here $a = 2$, $b = -3$, $c = -2$.

$$\Delta = (-3)^2 - 4(2)(-2) = 9 + 16 = 25$$

$$x = \frac{3 \pm \sqrt{25}}{4} = \frac{3 \pm 5}{4}$$

$$\text{So } x = \frac{3+5}{4} = 2 \text{ or } x = \frac{3-5}{4} = -\frac{1}{2}.$$

6.1.3 Completing the Square

The quadratic formula is actually derived from a technique called completing the square, which transforms any quadratic into a perfect square plus a constant:

$$ax^2 + bx + c = 0 \tag{6.1}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \tag{6.2}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \tag{6.3}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{6.4}$$

Taking the square root of both sides gives the quadratic formula.

6.2 Polynomial Equations

Quadratics are degree-2 polynomials. The pattern continues:

Definition 6.2 (Polynomial Equation). A polynomial equation of degree n has the form:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \quad (a_n \neq 0)$$

Remark 6.1 (Fundamental Theorem of Algebra). Every polynomial equation of degree n has exactly n solutions (counting multiplicities and complex numbers). This profound theorem was first proved by Carl Friedrich Gauss in his doctoral dissertation in 1799.

Some notable facts:

- Cubic equations (degree 3) can always be solved using a formula discovered by Cardano in 1545.

- Quartic equations (degree 4) can also be solved by formula (Ferrari, 1540s).
- Quintic equations (degree 5 and higher) *cannot* in general be solved by a formula involving only radicals. This was proved by Abel and Galois in the early 1800s—one of the most beautiful results in all of mathematics.

6.3 What Comes Next?

The journey you have begun in this book is just the first step. Here is a roadmap of the mathematical landscape that awaits:

1. Quadratic equations and functions: Parabolas, vertex form, optimization problems.
2. Polynomials and rational expressions: Long division, factor theorem, partial fractions.
3. Exponential and logarithmic equations: Growth, decay, the natural base e .
4. Trigonometry: Sine, cosine, tangent—the mathematics of angles and waves.
5. Analytic geometry: Circles, ellipses, parabolas, and hyperbolas (conic sections).
6. Calculus: The mathematics of change. Derivatives, integrals, and the fundamental theorem that connects them.
7. Linear algebra: Matrices, vectors, transformations—the backbone of computer graphics, machine learning, and quantum mechanics.
8. Number theory: Prime numbers, divisibility, modular arithmetic—the foundation of modern cryptography.
9. Probability and statistics: Uncertainty, distributions, hypothesis testing—essential for data science.
10. Abstract algebra and beyond: Groups, rings, fields—the deep structure of mathematics itself.

The most important thing: Mathematics is not a collection of disconnected topics. It is a *web of ideas*, where every concept connects to many others. The logic you learned in Chapter 2 underlies every proof. The equation-solving techniques from Chapters 4 and 5 are used in every branch of applied mathematics. The significant figures from Chapter 3 matter whenever mathematics meets the real world.

Keep going. The deeper you explore, the more beautiful it gets.

Remark 6.2. As the great mathematician Carl Friedrich Gauss once said: “Mathematics is the queen of the sciences and number theory is the queen of mathematics.” Whatever direction your studies take you, the foundations laid in this book will serve you well.

Further Reading

- [1] Polya, G. *How to Solve It*. Princeton University Press, 1945. (A classic guide to mathematical problem-solving.)
- [2] Velleman, D.J. *How to Prove It: A Structured Approach*, 3rd ed. Cambridge University Press, 2019. (Excellent introduction to mathematical logic and proof.)
- [3] Stewart, I. *Letters to a Young Mathematician*. Basic Books, 2006. (Inspiring and accessible.)
- [4] Lang, S. *Basic Mathematics*. Springer, 1988. (Comprehensive review of pre-calculus mathematics.)
- [5] Tiwari, G. “A Trip to Mathematics” series and other articles. <https://gauravtiwari.org>. (The original articles that inspired this book.)
- [6] Courant, R. and Robbins, H. *What Is Mathematics?* Oxford University Press, 1941. (A timeless exploration of fundamental ideas.)

Document typeset in $\text{\LaTeX}$ using STIX Two fonts.

February 2026

<https://gauravtiwari.org>