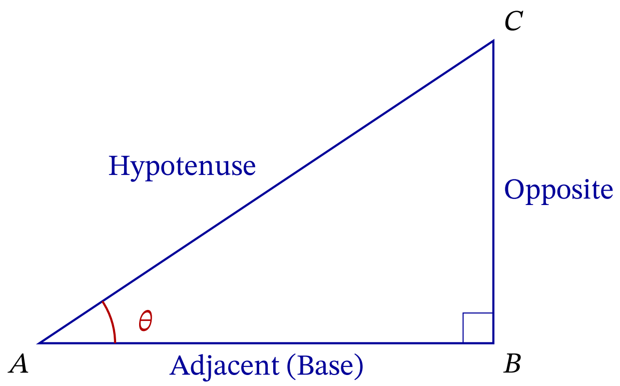
Trigonometric Identities

A Comprehensive Guide from Basics to Advanced

With Worked Examples and Practice Problems



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Contents

Part I

Foundations of Trigonometry

Chapter 1

Introduction to Trigonometry

1.1 What is Trigonometry?

Trigonometry is a branch of mathematics that studies the relationships between the sides and angles of triangles. The word "trigonometry" comes from the Greek words *trigonon* (triangle) and *metron* (measure), literally meaning "triangle measurement."

While trigonometry originated from the study of triangles, its applications extend far beyond geometry. Today, trigonometric functions are essential tools in:

- Physics (wave motion, oscillations, optics)
- Engineering (signal processing, electrical circuits)
- Astronomy (calculating distances to stars and planets)
- Navigation (GPS systems, surveying)
- Computer graphics (rotations, animations)
- Music theory (sound waves, harmonics)

1.2 What are Trigonometric Identities?

Definition

A trigonometric identity is an equation involving trigonometric functions that is true for all values of the variables for which both sides of the equation are defined.

Unlike trigonometric equations (which are true only for specific values), identities hold universally. For example:

- $\sin^2\theta + \cos^2\theta = 1$ is an identity (true for all θ)
- $\sin\theta = \frac{1}{2}$ is an equation (true only when $\theta = 30°, 150°, \ldots$)

1.3 Why Study Trigonometric Identities?

Trigonometric identities serve several important purposes:

1. Simplification: Complex trigonometric expressions can be reduced to simpler forms
2. Problem Solving: Many calculus and physics problems require identity manipulation
3. Verification: Identities help verify solutions to trigonometric equations
4. Transformation: Converting between different forms of the same expression

Why Identities Matter

Consider computing $\sin 75°$ without a calculator. Using the sum identity:

$$\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

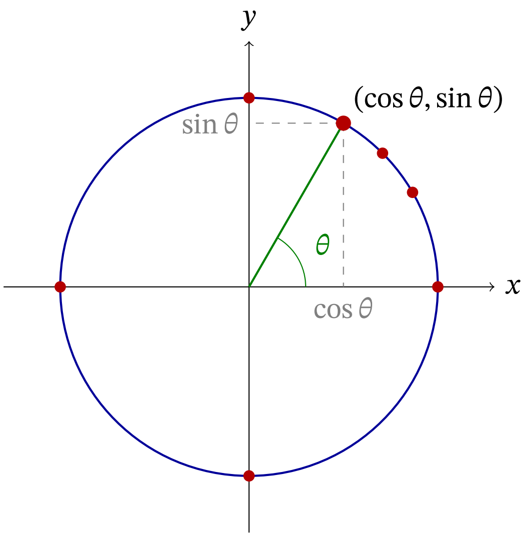
1.4 The Unit Circle: Foundation of Trigonometry

The unit circle provides a powerful way to understand trigonometric functions for all angles, not just those in right triangles.

Unit Circle

The unit circle is a circle with radius 1, centered at the origin of a coordinate plane. For any angle θ measured from the positive x-axis:

- $\cos\theta = x$-coordinate of the point on the circle
- $\sin\theta = y$-coordinate of the point on the circle



1.5 Angle Measurement: Degrees and Radians

Angles can be measured in two main units:

Degrees and Radians

Degrees: A full rotation is $360°$
Radians: A full rotation is 2π radians
Conversion: $180° = \pi$ radians, so:

$$1° = \frac{\pi}{180}\text{ radians} \qquad 1\text{ radian} = \frac{180°}{\pi} \approx 57.3°$$

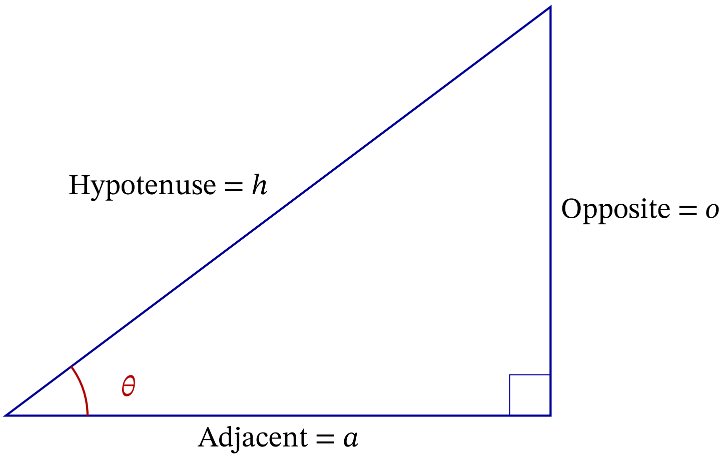
Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Chapter 2

The Six Trigonometric Functions

2.1 Definitions from Right Triangles

For a right-angled triangle with angle θ, the six trigonometric ratios are defined as:



2.1.1 Primary Trigonometric Functions

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{o}{h} \tag{2.1}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{a}{h} \tag{2.2}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{o}{a} \tag{2.3}$$

Note

Memory Aid: SOH-CAH-TOA

- Sine = Opposite / Hypotenuse
- Cosine = Adjacent / Hypotenuse
- Tangent = Opposite / Adjacent

2.1.2 Reciprocal Trigonometric Functions

$$\csc\theta = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{h}{o} = \frac{1}{\sin\theta} \tag{2.4}$$

$$\sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{h}{a} = \frac{1}{\cos\theta} \tag{2.5}$$

$$\cot\theta = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{a}{o} = \frac{1}{\tan\theta} \tag{2.6}$$

2.2 Values at Special Angles

Memorizing the values of trigonometric functions at special angles is essential.

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined
$\csc\theta$	undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
$\sec\theta$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	undefined
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Pattern for Sine Values

Notice the pattern in sine values at $0°, 30°, 45°, 60°, 90°$:

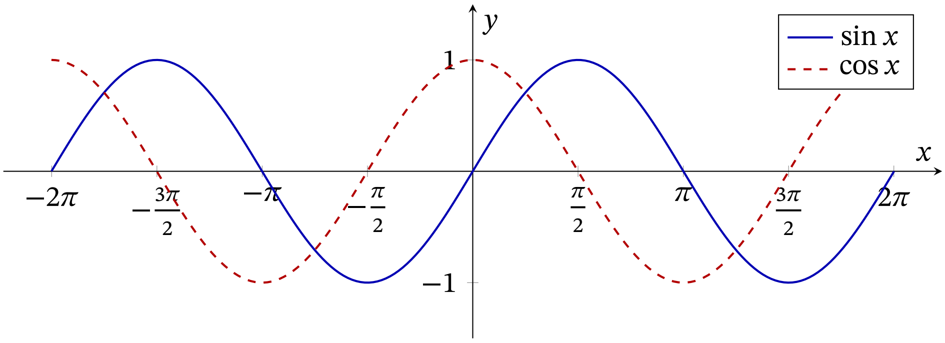
$$\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$$

Cosine follows the same pattern in reverse order.

2.3 Graphs of Trigonometric Functions

Understanding the graphs helps visualize the behavior of these functions.

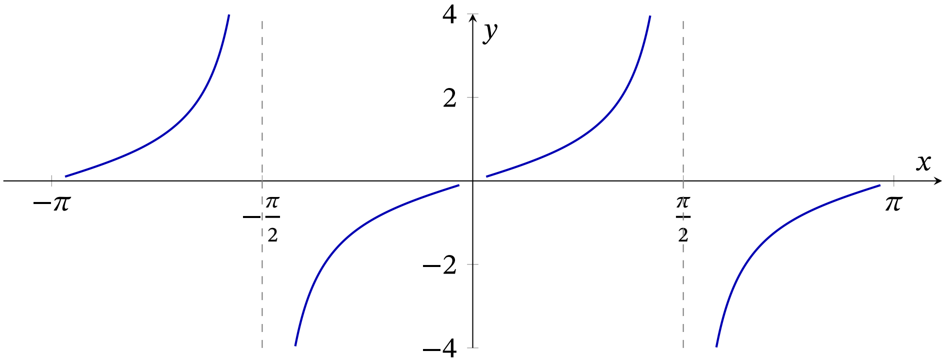
2.3.1 Sine and Cosine Functions



Key properties:

- Period: Both have period 2π
- Amplitude: Both have amplitude 1
- Range: $[-1, 1]$
- Phase shift: $\cos x = \sin(x + \frac{\pi}{2})$

2.3.2 Tangent and Cotangent Functions



Key properties of $\tan x$:

- Period: π
- Range: $(-\infty, \infty)$
- Vertical asymptotes: at $x = \frac{\pi}{2} + n\pi$ for integer n

Practice Problems

1. Find the exact value of $\sin 120°$ and $\cos 120°$.
2. If $\sin\theta = \frac{3}{5}$ and θ is in Quadrant I, find $\cos\theta$ and $\tan\theta$.
3. Sketch the graph of $y = 2\sin(x - \frac{\pi}{4})$.

Part II

Fundamental Identities

Chapter 3

Reciprocal and Ratio Identities

3.1 Reciprocal Identities

These identities express each trigonometric function in terms of its reciprocal.

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta} \tag{3.1}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \tag{3.2}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta} \tag{3.3}$$

Using Reciprocal Identities

Problem: If $\sin\theta = \frac{2}{3}$, find $\csc\theta$.
Solution: Using the reciprocal identity:

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

3.2 Ratio Identities

These identities express tangent and cotangent in terms of sine and cosine.

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{3.4}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{3.5}$$

Derivation: From the definitions in a right triangle:

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\frac{\text{Opposite}}{\text{Hypotenuse}}}{\frac{\text{Adjacent}}{\text{Hypotenuse}}} = \frac{\sin\theta}{\cos\theta}$$

Using Ratio Identities

Problem: If $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$, find $\tan\theta$ and $\cot\theta$.
Solution:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

Chapter 4

Pythagorean Identities

The Pythagorean identities are among the most important and frequently used trigonometric identities.

4.1 The Three Pythagorean Identities

Pythagorean Identities

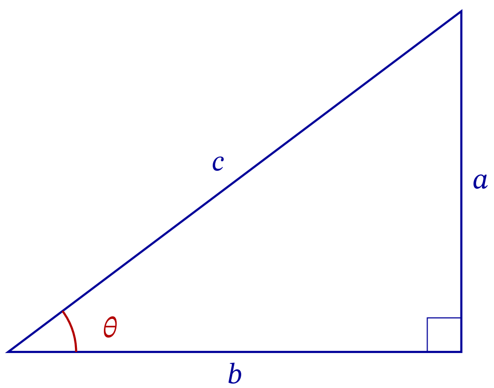
$$\sin^2\theta + \cos^2\theta = 1 \tag{4.1}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{4.2}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{4.3}$$

4.2 Proof of the First Pythagorean Identity

Consider a right-angled triangle with hypotenuse c, opposite side a, and adjacent side b.



By the Pythagorean theorem: $a^2 + b^2 = c^2$

Dividing both sides by c^2:

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

Since $\sin\theta = \frac{a}{c}$ and $\cos\theta = \frac{b}{c}$:

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

4.3 Derivation of the Other Pythagorean Identities

4.3.1 Second Identity: $1 + \tan^2\theta = \sec^2\theta$

Starting from $\sin^2\theta + \cos^2\theta = 1$, divide both sides by $\cos^2\theta$:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

4.3.2 Third Identity: $1 + \cot^2\theta = \csc^2\theta$

Starting from $\sin^2\theta + \cos^2\theta = 1$, divide both sides by $\sin^2\theta$:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

4.4 Alternate Forms

Each Pythagorean identity can be rearranged into useful alternate forms:

Original	Alternate Form 1	Alternate Form 2
$\sin^2\theta + \cos^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$	$\cos^2\theta = 1 - \sin^2\theta$
$1 + \tan^2\theta = \sec^2\theta$	$\tan^2\theta = \sec^2\theta - 1$	$\sec^2\theta - \tan^2\theta = 1$
$1 + \cot^2\theta = \csc^2\theta$	$\cot^2\theta = \csc^2\theta - 1$	$\csc^2\theta - \cot^2\theta = 1$

Worked Example 1

Problem: Simplify $\frac{\sin^2\theta}{\cos^2\theta} + 1$

Solution:

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \tan^2\theta + 1 = \sec^2\theta$$

Worked Example 2

Problem: If $\sin\theta = \frac{5}{13}$ and θ is in Quadrant I, find $\cos\theta$.
Solution: Using $\sin^2\theta + \cos^2\theta = 1$:

$$\left(\frac{5}{13}\right)^2 + \cos^2\theta = 1$$

$$\frac{25}{169} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{25}{169} = \frac{144}{169}$$

$$\cos\theta = \pm\frac{12}{13}$$

Since θ is in Quadrant I, $\cos\theta > 0$, so $\cos\theta = \frac{12}{13}$.

Worked Example 3

Problem: Prove that $\tan^2\theta - \sin^2\theta = \tan^2\theta \cdot \sin^2\theta$
Solution: Starting with the left side:

$$\begin{aligned}
\tan^2\theta - \sin^2\theta &= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta \\
&= \sin^2\theta\left(\frac{1}{\cos^2\theta} - 1\right) \\
&= \sin^2\theta\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) \\
&= \sin^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta}\right) \\
&= \sin^2\theta \cdot \tan^2\theta
\end{aligned}$$

This equals the right side, so the identity is verified.

Practice Problems

1. Simplify: $\sec^2\theta - 1$
2. If $\cos\theta = -\frac{3}{5}$ and θ is in Quadrant III, find $\sin\theta$ and $\tan\theta$.
3. Prove: $\frac{1-\sin^2\theta}{\csc^2\theta - 1} = \sin^2\theta$
4. Simplify: $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)$

Chapter 5

Even-Odd Identities

5.1 Definitions

Even and Odd Functions

A function $f(x)$ is:
- Even if $f(-x) = f(x)$ for all x (symmetric about y-axis)
- Odd if $f(-x) = -f(x)$ for all x (symmetric about origin)

5.2 Even-Odd Identities

Even-Odd Identities

Cosine and Secant are EVEN:

$$\cos(-\theta) = \cos\theta \tag{5.1}$$

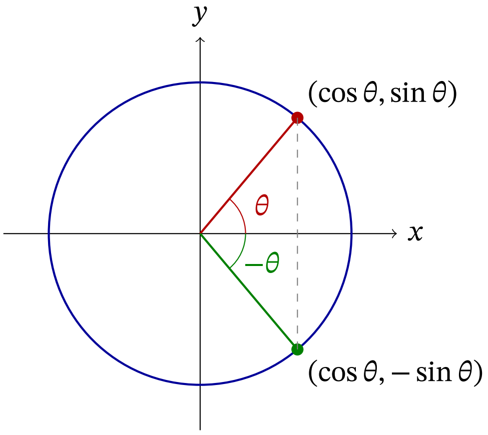
$$\sec(-\theta) = \sec\theta \tag{5.2}$$

Sine, Cosecant, Tangent, and Cotangent are ODD:

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta \tag{5.3}$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta \tag{5.4}$$

5.3 Understanding Through the Unit Circle



From the unit circle, we can see that for angle θ and angle $-\theta$:

- The x-coordinates are the same: $\cos(-\theta) = \cos\theta$ (even)
- The y-coordinates are opposite: $\sin(-\theta) = -\sin\theta$ (odd)

Using Even-Odd Identities

Problem: Simplify $\sin(-30^\circ) + \cos(-60^\circ)$
Solution:

$$\begin{aligned}\sin(-30^\circ) + \cos(-60^\circ) &= -\sin 30^\circ + \cos 60^\circ \\ &= -\frac{1}{2} + \frac{1}{2} = 0\end{aligned}$$

Part III

Sum, Difference, and Multiple Angle Identities

Chapter 6

Sum and Difference Identities

These identities allow us to find the trigonometric values of sums or differences of angles.

6.1 Sum and Difference Formulas for Sine

Sine Sum and Difference

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \tag{6.1}$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \tag{6.2}$$

6.1.1 Proof of Sine Sum Formula

Consider two angles A and B on the unit circle. Using the coordinates:

- Point at angle A: $(\cos A, \sin A)$
- Point at angle B: $(\cos B, \sin B)$

Using the rotation matrix or geometric arguments, we can show that:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

The difference formula follows by replacing B with $-B$:

$$\sin(A - B) = \sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$= \sin A \cos B - \cos A \sin B$$

6.2 Sum and Difference Formulas for Cosine

Cosine Sum and Difference

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \tag{6.3}$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \tag{6.4}$$

Note

Notice the sign pattern:

- In sine formulas, the sign between terms matches the operation
- In cosine formulas, the sign between terms is opposite to the operation

6.3 Sum and Difference Formulas for Tangent

Tangent Sum and Difference

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \tag{6.5}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \tag{6.6}$$

6.3.1 Derivation

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Dividing numerator and denominator by $\cos A \cos B$:

$$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

6.4 Worked Examples

Example 1: Finding Exact Values

Problem: Find the exact value of $\sin 75°$.
Solution: Write $75° = 45° + 30°$

$$\begin{aligned}\sin 75° &= \sin(45° + 30°)\\ &= \sin 45° \cos 30° + \cos 45° \sin 30°\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Example 2: Finding Exact Values

Problem: Find the exact value of $\cos 15°$.
Solution: Write $15° = 45° - 30°$

$$\begin{aligned}\cos 15° &= \cos(45° - 30°)\\ &= \cos 45° \cos 30° + \sin 45° \sin 30°\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Example 3: Using Given Information

Problem: If $\sin A = \frac{3}{5}$ (Quadrant I) and $\cos B = \frac{12}{13}$ (Quadrant I), find $\sin(A + B)$.
Solution: First, find the missing values:

- $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$
- $\sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \frac{144}{169}} = \frac{5}{13}$

Now apply the formula:

$$\begin{aligned}\sin(A + B) &= \sin A \cos B + \cos A \sin B\\ &= \frac{3}{5}\cdot\frac{12}{13} + \frac{4}{5}\cdot\frac{5}{13}\\ &= \frac{36}{65} + \frac{20}{65} = \frac{56}{65}\end{aligned}$$

Example 4: Tangent Addition

Problem: If $\tan A = 2$ and $\tan B = 3$, find $\tan(A + B)$.
Solution:

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{2 + 3}{1 - (2)(3)} = \frac{5}{1 - 6} = \frac{5}{-5} = -1$$

Practice Problems

1. Find the exact value of $\sin 105°$.
2. Find the exact value of $\tan 75°$.
3. If $\sin A = \frac{8}{17}$ (Quadrant II) and $\cos B = -\frac{3}{5}$ (Quadrant III), find $\cos(A - B)$.
4. Prove: $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$

Chapter 7

Double Angle Identities

Double angle identities express trigonometric functions of 2θ in terms of functions of θ.

7.1 Double Angle Formulas

Double Angle Identities

Sine:

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{7.1}$$

Cosine (three forms):

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta & (7.2)\\ &= 2\cos^2\theta - 1 & (7.3)\\ &= 1 - 2\sin^2\theta & (7.4) \end{aligned}$$

Tangent:

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{7.5}$$

7.2 Derivations

7.2.1 Sine Double Angle

Using the sum formula with $A = B = \theta$:

$$\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$$

7.2.2 Cosine Double Angle

Using the sum formula:

$$\cos 2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

The other forms come from substituting Pythagorean identities:

- Replace $\sin^2\theta = 1-\cos^2\theta$: $\cos 2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$

- Replace $\cos^2\theta = 1 - \sin^2\theta$: $\cos 2\theta = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$

7.3 Power Reduction Formulas

Rearranging the cosine double angle formulas gives power reduction formulas:

Power Reduction Formulas

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{7.6}$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \tag{7.7}$$

$$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta} \tag{7.8}$$

These are useful for integration and simplifying expressions involving squared trigonometric functions.

Example 1

Problem: If $\cos\theta = \frac{3}{5}$ and θ is in Quadrant IV, find $\sin 2\theta$.
Solution: First find $\sin\theta$. Since θ is in Quadrant IV, $\sin\theta < 0$:

$$\sin\theta = -\sqrt{1 - \cos^2\theta} = -\sqrt{1 - \frac{9}{25}} = -\frac{4}{5}$$

Now use the double angle formula:

$$\sin 2\theta = 2\sin\theta\cos\theta = 2 \cdot \left(-\frac{4}{5}\right) \cdot \frac{3}{5} = -\frac{24}{25}$$

Example 2

Problem: Simplify $\cos^4 x$ using power reduction formulas.
Solution:

$$\begin{aligned}
\cos^4 x = (\cos^2 x)^2 &= \left(\frac{1 + \cos 2x}{2}\right)^2 \\
&= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x) \\
&= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1 + \cos 4x}{2}\right) \\
&= \frac{1}{4} + \frac{\cos 2x}{2} + \frac{1}{8} + \frac{\cos 4x}{8} \\
&= \frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8}
\end{aligned}$$

Chapter 8

Half Angle Identities

8.1 Half Angle Formulas

Half Angle Identities

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \tag{8.1}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \tag{8.2}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta} \tag{8.3}$$

The $\pm$ sign is determined by the quadrant in which $\frac{\theta}{2}$ lies.

8.2 Derivation

Starting from the power reduction formula:

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Similarly for sine.

Example: Finding Exact Values

Problem: Find the exact value of $\sin 22.5°$.
Solution: Note that $22.5° = \frac{45°}{2}$, and $22.5°$ is in Quadrant I (so sine is positive).

$$\sin 22.5° = \sqrt{\frac{1-\cos 45°}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

Chapter 9

Triple Angle Identities

9.1 Triple Angle Formulas

Triple Angle Identities

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \tag{9.1}$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \tag{9.2}$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \tag{9.3}$$

9.2 Derivation of $\sin 3\theta$

$$\begin{aligned}
\sin 3\theta &= \sin(2\theta + \theta) \\
&= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\
&= 2\sin\theta\cos\theta\cdot\cos\theta + (1 - 2\sin^2\theta)\sin\theta \\
&= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta \\
&= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta \\
&= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta \\
&= 3\sin\theta - 4\sin^3\theta
\end{aligned}$$

Part IV

Product and Sum Identities

Chapter 10

Product-to-Sum Identities

These identities convert products of trigonometric functions into sums.

Product-to-Sum Formulas

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)] \tag{10.1}$$

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)] \tag{10.2}$$

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)] \tag{10.3}$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)] \tag{10.4}$$

10.1 Derivation

Adding and subtracting the sum and difference formulas:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Adding: $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$

Therefore: $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

Example

Problem: Express $\sin 5x \cos 3x$ as a sum.
Solution:

$$\sin 5x \cos 3x = \frac{1}{2}[\sin(5x+3x) + \sin(5x-3x)] = \frac{1}{2}[\sin 8x + \sin 2x]$$

Chapter 11

Sum-to-Product Identities

These identities convert sums of trigonometric functions into products.

Sum-to-Product Formulas

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \tag{11.1}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \tag{11.2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \tag{11.3}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2} \tag{11.4}$$

Example

Problem: Express $\cos 5x - \cos 3x$ as a product.
Solution:

$$\cos 5x - \cos 3x = -2\sin\frac{5x+3x}{2}\sin\frac{5x-3x}{2} = -2\sin 4x\sin x$$

Part V

Cofunction and Periodicity Identities

Chapter 12

Cofunction Identities

12.1 Complementary Angles

Definition

Two angles are complementary if their sum equals 90° (or $\frac{\pi}{2}$ radians).

Cofunction Identities

$$\sin(90° - \theta) = \cos\theta \qquad \cos(90° - \theta) = \sin\theta \tag{12.1}$$

$$\tan(90° - \theta) = \cot\theta \qquad \cot(90° - \theta) = \tan\theta \tag{12.2}$$

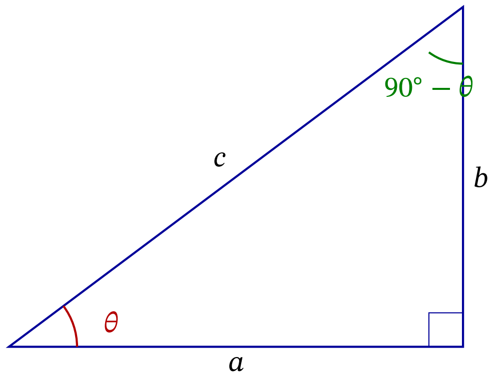
$$\sec(90° - \theta) = \csc\theta \qquad \csc(90° - \theta) = \sec\theta \tag{12.3}$$

In radians:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \tag{12.4}$$

12.2 Geometric Interpretation

In a right triangle with acute angles θ and $90° - \theta$:



The opposite side for angle θ is the adjacent side for angle $90° - \theta$, and vice versa. This explains why:

$$\sin\theta = \frac{b}{c} = \cos(90° - \theta)$$

12.3 Supplementary Angles

Definition

Two angles are supplementary if their sum equals 180° (or π radians).

Supplementary Angle Identities

$$\sin(180° - \theta) = \sin\theta \qquad \csc(180° - \theta) = \csc\theta \tag{12.5}$$

$$\cos(180° - \theta) = -\cos\theta \qquad \sec(180° - \theta) = -\sec\theta \tag{12.6}$$

$$\tan(180° - \theta) = -\tan\theta \qquad \cot(180° - \theta) = -\cot\theta \tag{12.7}$$

Chapter 13

Periodicity Identities

13.1 Period of Trigonometric Functions

Definition

A function $f(x)$ is periodic with period T if $f(x+T) = f(x)$ for all x.

Function	Period
$\sin x$, $\cos x$, $\csc x$, $\sec x$	2π
$\tan x$, $\cot x$	π

13.2 Periodicity Identities

Periodicity Identities (Radians)

Adding $\frac{\pi}{2}$:

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta \tag{13.1}$$

Adding π:

$$\sin(\pi + \theta) = -\sin\theta \qquad \cos(\pi + \theta) = -\cos\theta \tag{13.2}$$

$$\tan(\pi + \theta) = \tan\theta \tag{13.3}$$

Adding $\frac{3\pi}{2}$:

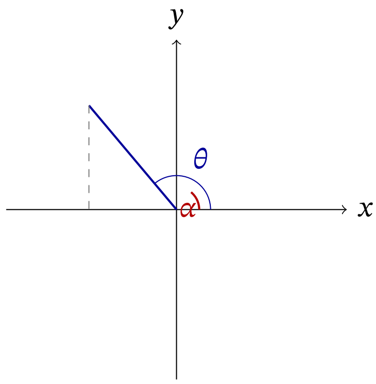
$$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta \qquad \cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta \tag{13.4}$$

Adding 2π:

$$\sin(2\pi + \theta) = \sin\theta \qquad \cos(2\pi + \theta) = \cos\theta \tag{13.5}$$

13.3 Reference Angle Method

The reference angle is the acute angle that the terminal side makes with the x-axis.



Quadrant II: $\alpha = 180° - \theta$

Reference Angle Rules

For angle θ:

- Quadrant I: Reference angle $= \theta$
- Quadrant II: Reference angle $= 180° - \theta$
- Quadrant III: Reference angle $= \theta - 180°$
- Quadrant IV: Reference angle $= 360° - \theta$

Part VI

Triangle Identities and Laws

Chapter 14

Law of Sines

The Law of Sines applies to any triangle, not just right triangles.

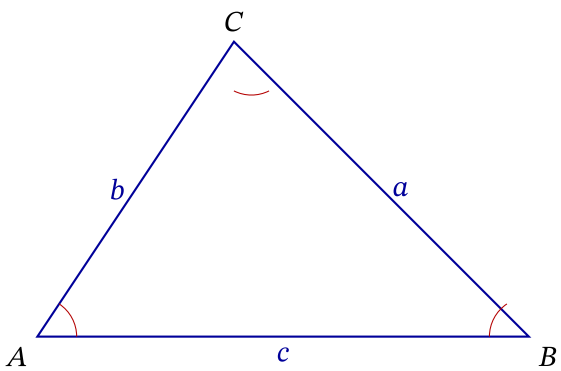
Law of Sines

In any triangle with sides a, b, c opposite to angles A, B, C respectively:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Or equivalently:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



14.1 Applications

The Law of Sines is useful for:

1. AAS (Angle-Angle-Side): Given two angles and a non-included side
2. ASA (Angle-Side-Angle): Given two angles and the included side
3. SSA (Side-Side-Angle): Given two sides and an angle opposite one of them (ambiguous case)

Example

Problem: In triangle ABC, $A = 40°$, $B = 60°$, and $a = 8$. Find b.
Solution: Using the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{8}{\sin 40°} = \frac{b}{\sin 60°}$$

$$b = \frac{8 \sin 60°}{\sin 40°} = \frac{8 \cdot \frac{\sqrt{3}}{2}}{0.6428} \approx 10.78$$

Chapter 15

Law of Cosines

Law of Cosines

In any triangle with sides a, b, c opposite to angles A, B, C:

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{15.1}$$
$$b^2 = a^2 + c^2 - 2ac\cos B \tag{15.2}$$
$$c^2 = a^2 + b^2 - 2ab\cos C \tag{15.3}$$

Note

When $C = 90°$, $\cos C = 0$, and the Law of Cosines reduces to the Pythagorean theorem: $c^2 = a^2 + b^2$.

15.1 Solving for Angles

Rearranging to find angles:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

15.2 Applications

The Law of Cosines is useful for:

1. SAS (Side-Angle-Side): Given two sides and the included angle
2. SSS (Side-Side-Side): Given all three sides

Example: SAS

Problem: In triangle ABC, $b = 5$, $c = 7$, and $A = 60°$. Find a.
Solution:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 25 + 49 - 2(5)(7)\cos 60° \\ &= 74 - 70 \cdot \frac{1}{2} \\ &= 74 - 35 = 39 \\ a &= \sqrt{39} \approx 6.24 \end{aligned}$$

Chapter 16

Law of Tangents

Law of Tangents

In any triangle:

$$\frac{a-b}{a+b} = \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}$$

This identity is less commonly used than the Laws of Sines and Cosines, but can be useful in certain computational situations.

Chapter 17

Area Formulas for Triangles

17.1 Basic Area Formula

Area using Sine

The area of a triangle with sides a and b and included angle C:

$$\text{Area} = \frac{1}{2}ab\sin C$$

17.2 Heron's Formula

Heron's Formula

For a triangle with sides a, b, c and semi-perimeter $s = \frac{a+b+c}{2}$:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

Example

Problem: Find the area of a triangle with sides 7, 8, and 9.
Solution:

$$s = \frac{7+8+9}{2} = 12$$

$$\text{Area} = \sqrt{12(12-7)(12-8)(12-9)} = \sqrt{12 \cdot 5 \cdot 4 \cdot 3} = \sqrt{720} = 12\sqrt{5} \approx 26.83$$

Part VII

Advanced Topics

Chapter 18

Inverse Trigonometric Functions

18.1 Definitions

Inverse Trigonometric Functions

$$y = \arcsin x \quad \Leftrightarrow \quad x = \sin y, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \tag{18.1}$$

$$y = \arccos x \quad \Leftrightarrow \quad x = \cos y, \quad 0 \leq y \leq \pi \tag{18.2}$$

$$y = \arctan x \quad \Leftrightarrow \quad x = \tan y, \quad -\frac{\pi}{2} < y < \frac{\pi}{2} \tag{18.3}$$

18.2 Properties

Inverse Function Properties

For appropriate domains:

$$\sin(\arcsin x) = x \qquad \arcsin(\sin x) = x \tag{18.4}$$

$$\cos(\arccos x) = x \qquad \arccos(\cos x) = x \tag{18.5}$$

$$\tan(\arctan x) = x \qquad \arctan(\tan x) = x \tag{18.6}$$

18.3 Identities Involving Inverse Functions

Sum of Inverse Functions

$$\arcsin x + \arccos x = \frac{\pi}{2} \tag{18.7}$$

$$\arctan x + \arctan \frac{1}{x} = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases} \tag{18.8}$$

Chapter 19

Hyperbolic Functions

While not strictly trigonometric, hyperbolic functions have similar identities.

19.1 Definitions

Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \tag{19.1}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x} \tag{19.2}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x} \tag{19.3}$$

19.2 Hyperbolic Identities

Fundamental Hyperbolic Identity

$$\cosh^2 x - \sinh^2 x = 1$$

Compare with: $\cos^2 x + \sin^2 x = 1$

Other Hyperbolic Identities

$$1 - \tanh^2 x = \operatorname{sech}^2 x \tag{19.4}$$

$$\coth^2 x - 1 = \operatorname{csch}^2 x \tag{19.5}$$

$$\sinh 2x = 2 \sinh x \cosh x \tag{19.6}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x \tag{19.7}$$

Chapter 20

Euler's Formula and Complex Numbers

20.1 Euler's Formula

Euler's Formula

For any real number θ:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where $i = \sqrt{-1}$ is the imaginary unit.

20.2 Consequences

Setting $\theta = \pi$:

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0 = -1$$

This gives Euler's Identity:

$$\boxed{e^{i\pi} + 1 = 0}$$

Often called "the most beautiful equation in mathematics."

20.3 Deriving Identities from Euler's Formula

From $e^{i\theta} = \cos\theta + i\sin\theta$, we can derive:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

De Moivre's Theorem:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

This can be used to derive multiple angle formulas.

Part VIII

Applications and Practice

Chapter 21

Proving Trigonometric Identities

21.1 Strategies for Proofs

1. Work on one side: Transform one side to match the other
2. Work on both sides: Simplify both sides to a common form
3. Convert to sines and cosines: Often simplifies complex expressions
4. Factor: Look for common factors or factorable expressions
5. Use conjugates: Multiply by $\frac{1+\sin x}{1+\sin x}$ or similar

21.2 Worked Examples

Example 1

Prove: $\frac{1+\tan^2\theta}{\tan^2\theta} = \csc^2\theta$
Proof: Starting with the left side:

$$\begin{aligned}
\frac{1+\tan^2\theta}{\tan^2\theta} &= \frac{\sec^2\theta}{\tan^2\theta} \quad \text{(Pythagorean identity)}\\
&= \frac{\frac{1}{\cos^2\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\\
&= \frac{1}{\sin^2\theta}\\
&= \csc^2\theta
\end{aligned}$$

Example 2

Prove: $\frac{\sin\theta}{1-\cos\theta} = \csc\theta + \cot\theta$
Proof: Multiply numerator and denominator by $1+\cos\theta$:

$$\begin{aligned}\frac{\sin\theta}{1-\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} &= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} \\ &= \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} \\ &= \frac{1+\cos\theta}{\sin\theta} \\ &= \frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta} \\ &= \csc\theta+\cot\theta\end{aligned}$$

Example 3

Prove: $\cos^4\theta - \sin^4\theta = \cos 2\theta$
Proof: Factor as difference of squares:

$$\begin{aligned}\cos^4\theta - \sin^4\theta &= (\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta) \\ &= (1)(\cos 2\theta) \\ &= \cos 2\theta\end{aligned}$$

Chapter 22

Applications in Calculus

22.1 Derivatives

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

22.2 Integrals

Function	Integral
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\tan x$	$-\ln\lvert\cos x\rvert + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$

22.3 Trigonometric Substitutions

Power reduction formulas are essential for integrating powers of trigonometric functions:

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

Chapter 23

Real-World Applications

23.1 Wave Motion

Waves can be described using trigonometric functions:

$$y(x, t) = A \sin(kx - \omega t + \phi)$$

where:

- A = amplitude
- k = wave number
- ω = angular frequency
- ϕ = phase constant

23.2 Simple Harmonic Motion

A mass on a spring oscillates according to:

$$x(t) = A \cos(\omega t + \phi)$$

23.3 AC Circuits

Alternating current and voltage are sinusoidal:

$$V(t) = V_0 \sin(\omega t)$$

$$I(t) = I_0 \sin(\omega t + \phi)$$

The phase difference ϕ depends on the circuit components (resistors, capacitors, inductors).

Quick Reference Tables

Fundamental Identities

Category	Identities
Pythagorean	$\sin^2\theta + \cos^2\theta = 1$ $1 + \tan^2\theta = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$
Reciprocal	$\csc\theta = \frac{1}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta}, \cot\theta = \frac{1}{\tan\theta}$
Ratio	$\tan\theta = \frac{\sin\theta}{\cos\theta}, \cot\theta = \frac{\cos\theta}{\sin\theta}$
Even-Odd	$\cos(-\theta) = \cos\theta, \sin(-\theta) = -\sin\theta$

Sum and Difference Formulas

Function	Sum	Difference
sin	$\sin A\cos B + \cos A\sin B$	$\sin A\cos B - \cos A\sin B$
cos	$\cos A\cos B - \sin A\sin B$	$\cos A\cos B + \sin A\sin B$
tan	$\frac{\tan A+\tan B}{1-\tan A\tan B}$	$\frac{\tan A-\tan B}{1+\tan A\tan B}$

Double Angle Formulas

$\sin 2\theta$	$2\sin\theta\cos\theta$
$\cos 2\theta$	$\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
$\tan 2\theta$	$\frac{2\tan\theta}{1-\tan^2\theta}$

Half Angle Formulas

$\sin\frac{\theta}{2}$	$\pm\sqrt{\frac{1-\cos\theta}{2}}$
$\cos\frac{\theta}{2}$	$\pm\sqrt{\frac{1+\cos\theta}{2}}$
$\tan\frac{\theta}{2}$	$\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$

Product-to-Sum Formulas

$$\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$$
$$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$
$$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

Sum-to-Product Formulas

$$\sin A+\sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A-\sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A+\cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A-\cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Triangle Laws

Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$

Area: $\text{Area} = \frac{1}{2}ab\sin C$

Special Angle Values

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.

Practice Problem Answers

Chapter 2 Answers

1. $\sin 120° = \frac{\sqrt{3}}{2}$, $\cos 120° = -\frac{1}{2}$
2. $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$
3. Graph has amplitude 2, phase shift $\frac{\pi}{4}$ right

Chapter 4 Answers

1. $\tan^2\theta$
2. $\sin\theta = -\frac{4}{5}$, $\tan\theta = \frac{4}{3}$
3. Proof uses $\sin^2\theta = 1 - \cos^2\theta$ and $\csc^2\theta - 1 = \cot^2\theta$
4. $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = \sec^2\theta - \tan^2\theta = 1$

Chapter 6 Answers

1. $\sin 105° = \frac{\sqrt{6}+\sqrt{2}}{4}$
2. $\tan 75° = 2 + \sqrt{3}$
3. $\cos(A - B) = \frac{36}{85}$ or $-\frac{84}{85}$ depending on quadrant
4. Use sum formula twice and add