

Function: Notations and Rules

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The domain can change whether a function is invertible even when its formula stays the same.

Consider $f(x) = x^2$. Over the real numbers, $f(2) = f(-2) = 4$, so the function is not one-to-one. Restricting its domain to $[0, \infty)$ gives a unique nonnegative input for each output. Its inverse is then $f^{-1}(y) = \sqrt{y}$, for $y \geq 0$.

TLDR;

- A function is a relation where every input maps to exactly one output. This page covers the formal definition, standard notation like $f(x)$, and the concepts of domain and range.
- You will learn the key types of functions: into, onto, one-to-one, restriction, and extension, each with concrete examples so you can identify them in practice.
- The guide also covers operations on functions (addition, subtraction, multiplication, division) and includes FAQs addressing the most common points of confusion.

What Is a Function?

A function in math is a special kind of relation where every input gives you exactly one output. If you pick any element from the domain and run it through the function, you always land on a single, definite value. There is no ambiguity, no branching into two different results.

Formally, f is a function if and only if two conditions hold:

1. Every member of f is an **ordered pair**. That is, $xfy \Rightarrow (x, y) \in f$.
2. If both $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. No input maps to two different outputs.

Here is a quick way to think about it. Consider the **set** $f = \{(1, 3), (2, 5), (3, 7)\}$. Each first coordinate (1, 2, 3) appears exactly once, so this is a valid function. But if you had $g = \{(1, 3), (1, 5), (2, 7)\}$, the input 1 maps to both 3 and 5. That violates the second condition, so g is not a function.

The vertical line test is the graphical version of the function definition. If any vertical line crosses a graph more than once, that graph doesn't represent a function. It's the quickest way to check whether a relation qualifies as a function in math.

Notation for Functions

When you write $(x, y) \in f$, you are saying that the ordered pair (x, y) belongs to the function f . Here, x is the **argument** (or input) and y is the **image** (or output). The older notation xfy expresses the same relationship, though you will rarely see it in modern textbooks.

The set of all valid arguments is called the **domain** of the function, and the set of all images (the actual outputs you get) is called the **range**. Keep these two terms sharp in your mind because they come up constantly.

Several equivalent notations exist for expressing $(x, y) \in f$:

- $y = f(x)$ is the standard modern notation you will use most often.
- $y = fx$ is a shorthand sometimes used in algebra and analysis.
- $y : xf$ and $y = x^f$ are older notational forms you may encounter in classical texts.

For example, if $f(x) = 2x+1$, then $f(3) = 7$. The argument is 3, the image is 7, and you write $(3, 7) \in f$. This notation is fundamental to working with **equations** across all branches of mathematics.

Domain and Range

The **domain** of a function f is the set of all inputs for which f is defined. The **range** (also called the image) is the set of all outputs that f actually produces. These are not the same as the codomain, which is the larger set that the range sits inside.

Consider $f(x) = x^2$ where $x \in \mathbb{R}$. The domain is all of $\mathbb{R}$, but the range is only $[0, \infty)$ because squaring a real number never gives a negative result. If you are told that $f : \mathbb{R} \rightarrow \mathbb{R}$, the codomain is $\mathbb{R}$, but the range is still just the non-negative reals.

Understanding this distinction is critical. When you encounter a function in math, your first question should always be: what is its domain, and what values does it actually take? If you need to find where a function equals zero, that's a separate but related concept covered in the **zero of a function** study note.

Into Function

A function f is said to be **into** Y if the range of f is a **subset** of Y , but does not necessarily cover all of Y . In set notation, $R_f \subset Y$. Some elements of Y are left "unhit" by the function.

For example, let $f : \{1, 2, 3\} \rightarrow \{a, b, c, d\}$ be defined by $f(1) = a$, $f(2) = b$, $f(3) = c$. The range is $\{a, b, c\}$, which is a proper subset of the codomain $\{a, b, c, d\}$. The element d has no preimage, so f is

an into function. You can think of it as a function that maps "into" the codomain without filling it completely.

Onto Function

A function f is **onto** Y (also called **surjective**) if the range of f equals Y exactly. In set notation, $R_f = Y$. Every element in the codomain is hit by at least one element from the domain.

For example, let $g : \{1, 2, 3, 4\} \rightarrow \{a, b, c\}$ be defined by $g(1) = a$, $g(2) = b$, $g(3) = c$, $g(4) = a$. The range is $\{a, b, c\}$, which equals the codomain. Every element in the codomain has at least one preimage, so g is onto. Notice that a is hit twice, that is perfectly fine for a surjection.

The general notation for a mapping is $f : X \rightarrow Y$, where X is the domain and Y is the codomain. Whether the function is into or onto depends on whether the range fills up all of Y .

One-to-One Function

A function is **one-to-one** (also called **injective**) if it maps distinct elements to distinct elements. No two different inputs ever produce the same output. Formally, f is one-to-one if and only if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. Equivalently, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Consider $f(x) = 3x + 2$ defined on $\mathbb{R}$. If $f(x_1) = f(x_2)$, then $3x_1 + 2 = 3x_2 + 2$, which gives $x_1 = x_2$. So f is one-to-one. Now consider $g(x) = x^2$ on $\mathbb{R}$. Here $g(2) = 4$ and $g(-2) = 4$, so two different inputs give the same output. This function is not one-to-one.

A function that is both one-to-one and onto is called a **bijection**. Bijections are especially important because they have inverses: you can uniquely reverse the mapping from output back to input.

Restriction of a Function

If you have a function $f : X \rightarrow Y$ and you want to limit its domain to a smaller set $A \subseteq X$, you get the **restriction** of f to A . Formally, the restriction is $f \cap (A \times Y)$, which is usually written as $f|_A$. It is still a function from A into Y , but it only "sees" the inputs in A .

For example, let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$. If you restrict f to $A = [0, \infty)$, you get $f|_{[0, \infty)}(x) = x^2$ for $x \geq 0$. This restricted version is actually one-to-one (since you have removed the negative inputs that caused duplicates), and it is a common technique for making a function invertible.

Extension of a Function

An **extension** goes in the opposite direction. A function f is an extension of a function g if $g \subseteq f$, meaning that f agrees with g everywhere that g is defined, but f may also be defined on additional inputs outside the domain of g .

For example, let $g : \{1, 2\} \rightarrow \mathbb{R}$ with $g(1) = 5$ and $g(2) = 8$. Now define $f : \{1, 2, 3\} \rightarrow \mathbb{R}$ with $f(1) = 5$, $f(2) = 8$, and $f(3) = 11$. Since f matches g on inputs 1 and 2 and adds a new mapping for 3, we say f is an extension of g . Conversely, g is the restriction of f to $\{1, 2\}$. These two concepts are mirror images of each other.

Operations on Functions

Once you have two functions defined on the same domain, you can combine them using arithmetic operations. If f and g are both functions from X to $\mathbb{R}$, you can define the following:

- **Addition:** $(f + g)(x) = f(x) + g(x)$
- **Subtraction:** $(f - g)(x) = f(x) - g(x)$
- **Multiplication:** $(f \cdot g)(x) = f(x) \cdot g(x)$
- **Division:** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$

The domain of the combined function is the intersection of the domains of f and g . For division, you must also exclude any x where $g(x) = 0$.

Function operations become much easier when you're comfortable with algebraic manipulation. If you need a refresher on solving and simplifying expressions, check out the [equations](#) study note before tackling these problems.

As a concrete example, let $f(x) = x + 1$ and $g(x) = x - 2$, both defined on $\mathbb{R}$. Then $(f + g)(x) = 2x - 1$, $(f \cdot g)(x) = (x + 1)(x - 2) = x^2 - x - 2$, and $\left(\frac{f}{g}\right)(x) = \frac{x+1}{x-2}$ for all $x \neq 2$.

Functions in Trigonometry and Calculus

The concept of a function in math extends far beyond simple algebraic expressions. Trigonometric functions like $\sin(x)$, $\cos(x)$, and $\tan(x)$ are some of the most widely used functions in higher mathematics. Each one maps a real number (an angle) to a specific output value, following the exact same rules covered above.

What makes trigonometric functions interesting is that they're periodic, meaning they repeat their outputs at regular intervals. This periodicity means they aren't one-to-one on their full domain,

which is why restrictions (covered earlier) become essential when defining inverse trig functions. For a deeper look at how these functions relate to each other, see the [trigonometric identities](#) study note.

In calculus, functions take center stage. **Derivatives** measure how a function's output changes relative to its input, and integrals measure the accumulation of a function's values over an interval. If you're heading toward calculus, the [best calculus books](#) can help you build a strong foundation in function analysis, limits, and continuity.

Key Takeaways

Here are the main ideas you should carry away from this page:

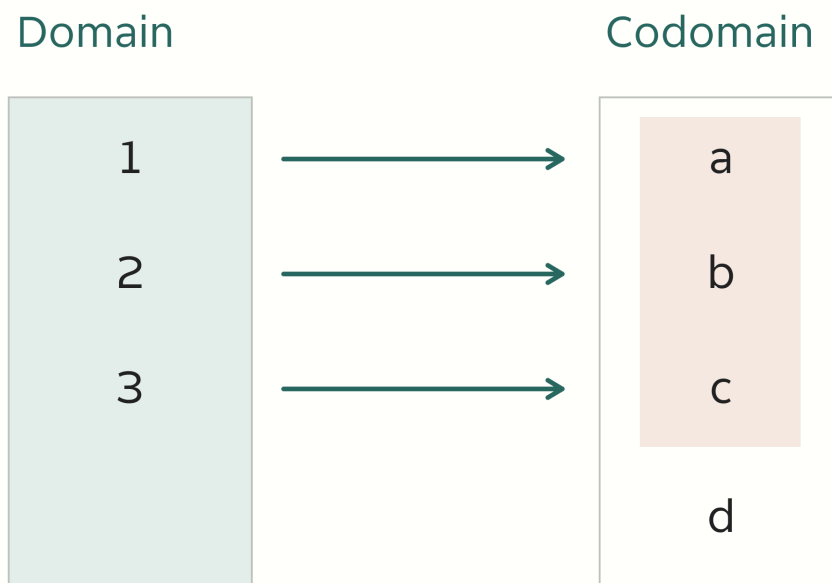
- A function maps each input to exactly one output. The uniqueness of the output is what distinguishes a function from a general relation.
- The standard notation $y = f(x)$ tells you that x is the input and y is the output. The domain is the set of valid inputs, and the range is the set of actual outputs.
- An **into** function does not cover its entire codomain. An **onto** (surjective) function does.
- A **one-to-one** (injective) function never sends two different inputs to the same output. A function that is both one-to-one and onto is a **bijection**.
- **Restriction** shrinks the domain. **Extension** expands it. These are inverse operations and are commonly used to make functions invertible or to define piecewise behavior.
- You can add, subtract, multiply, and divide functions, as long as you respect the domains. For division, always exclude points where the denominator is zero.

The related [OpenStax mathematics textbooks](#) provide additional explanations and exercises for the mathematical definitions used here.

Domain Restrictions and Inverses

For $g(x) = 1/(x - 3)$, the domain excludes 3. The value $g(3)$ is undefined; it is not a very large real number. A composition also needs a domain check: the inner output must belong to the outer domain. For $\sqrt{x - 2}$, this requires $x \geq 2$.

Domain, Codomain, Range



Range = {a, b, c}. The codomain also includes d.
Every input has exactly 1 output.

Function arrows connect domain values 1, 2 and 3 to a, b and c; codomain value d is outside the range.

The points to check in this example are:

- The range contains reached outputs; the codomain is the declared target set.
- A many-to-one rule is still a function because each input has a single output.
- An inverse function and the reciprocal $1/f(x)$ are different constructions.

Check Your Understanding

Try the questions before reading the answers.

Question 1. Is $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$, onto?

Answer. No. No negative real number is an output.

Question 2. What is $f^{-1}(9)$ on the restricted nonnegative domain?

Answer. 3. The restriction excludes the other square root, -3.

Practice Questions

Work each question before reading its solution. The set runs from direct recall and substitution to the applied questions that exams actually use to separate grades. The downloadable PDF includes these questions and their solutions.

Question 1. For $f(x) = 3x^2 - 2x + 1$, compute $f(2)$ and $f(-1)$.

Solution. $f(2) = 3(4) - 4 + 1 = 9$. $f(-1) = 3(1) + 2 + 1 = 6$. Substitute the input everywhere x appears, keeping the parentheses so signs survive: $(-1)^2 = 1$, not -1 .

Question 2. Why is $f(x)$ not "f times x"? What operation does the notation actually name?

Solution. f is not a number but a rule; $f(x)$ names the output of applying the rule f to the input x . Reading it as multiplication produces nonsense like canceling the parentheses. The notation is a machine label: input in, output out.

Question 3. For $f(x) = x^2 + 1$, expand $f(a + h)$, and explain the most common error.

Solution. $f(a + h) = (a + h)^2 + 1 = a^2 + 2ah + h^2 + 1$. The error is writing $f(a) + f(h)$ or $a^2 + h^2 + 1$: functions do not distribute over addition. The whole input is squared, cross term included; that cross term is exactly what makes derivatives work later.

Question 4. For $f(x) = 2x + 3$ and $g(x) = x^2$, compute $f(g(2))$ and $g(f(2))$.

Solution. Inside out: $g(2) = 4$, then $f(4) = 11$. The other order: $f(2) = 7$, then $g(7) = 49$. Composition is not commutative: $f \circ g$ and $g \circ f$ are different functions, and 11 versus 49 proves it on the spot.

Question 5. Find the domain of $f(x) = \frac{1}{x-3}$ and $g(x) = \sqrt{x+2}$.

Solution. f : all reals except 3, where the denominator vanishes. g : $x \geq -2$, so the radicand stays nonnegative. Domain questions reduce to 2 bans: no division by zero, no even root of a negative.

Question 6. For $f(x) = x^2$, describe what $f(x) + 2$, $f(x + 2)$, and $2f(x)$ each do to the graph.

Solution. $f(x) + 2$ shifts up 2. $f(x + 2)$ shifts LEFT 2, opposite to intuition, because inputs reach any given output 2 earlier. $2f(x)$ stretches vertically by 2. Outside the parentheses acts on outputs (vertical); inside acts on inputs (horizontal, reversed).

Question 7. Compute the difference quotient $\frac{f(x+h) - f(x)}{h}$ for $f(x) = x^2$.

Solution. $\frac{(x+h)^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h$. As $h \rightarrow 0$ this becomes $2x$, the derivative. The difference quotient is the bridge between function notation and calculus, and it works only if $f(x+h)$ was expanded correctly.

Question 8. Is $f(x) = x^2$ one-to-one? What does your answer mean for an inverse function?

Solution. No: $f(2) = f(-2) = 4$, two inputs sharing an output, so it fails the horizontal line test.

Without one-to-one there is no inverse on the full domain; restricting to $x \geq 0$ repairs it, giving $f^{-1}(x) = \sqrt{x}$. Inverses exist exactly when every output points back to one input.

Question 9. For $f(x) = \frac{x}{x-1}$, compute $f(f(x))$ and simplify.

Solution. $f(f(x)) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{\frac{x}{x-1}}{\frac{x - (x-1)}{x-1}} = \frac{x}{1} = x$. This function is its own inverse: applying it twice returns the input. Such involutions exist, and finding one by direct computation is a standard exam surprise.

Question 10. A cab charges a flat 50 plus 12 per km. Write the fare as a function, state $C(7)$, and interpret $C(0)$.

Solution. $C(d) = 50 + 12d$. $C(7) = 50 + 84 = 134$. $C(0) = 50$ is the flag-drop: the cost of entering the cab before it moves. Function notation packages a real pricing rule so any question about it becomes an evaluation.

Frequently Asked Questions

What is the difference between a function and a relation?

A relation is any set of ordered pairs. A function is a special type of relation where each input (first coordinate) appears in exactly one ordered pair. In other words, every input maps to one and only one output. The set $\{(1, 2), (1, 3)\}$ is a valid relation but not a function because the input 1 has two different outputs.

What is the difference between range and codomain?

The codomain is the set that you declare as the target of a function when you write $f : X \rightarrow Y$. The range (or image) is the subset of the codomain that the function actually hits. For example, if $f(x) = x^2$ maps from the reals to the reals, the codomain is all of $\mathbb{R}$, but the range is only $[0, \infty)$. The range is always a subset of the codomain, and they are equal only when the function is onto (surjective).

How can you tell if a function is one-to-one from its graph?

Use the horizontal line test. Draw horizontal lines across the graph. If every horizontal line intersects the graph at most once, the function is one-to-one. If any horizontal line crosses the graph more than once, there are two different inputs producing the same output, and the function fails to be injective. For example, $f(x) = x^3$ passes the horizontal line test, but $f(x) = x^2$ does not because horizontal lines above zero cross the parabola at two points.

Why would you restrict a function to a smaller domain?

The most common reason is to make a function invertible. A function must be one-to-one (injective) to have an inverse, and many standard functions like x^2 , $\sin(x)$, and $\cos(x)$ are not one-to-one on their full natural domain. By restricting x^2 to $[0, \infty)$, or $\sin(x)$ to $[-\pi/2, \pi/2]$, you eliminate the duplicate outputs and obtain a one-to-one function that can be properly inverted. This is exactly how inverse functions like $\sqrt{x}$ and $\arcsin(x)$ are defined.

What does it mean for a function to be a bijection?

A bijection is a function that is both one-to-one (injective) and onto (surjective). This means every output in the codomain is hit by exactly one input from the domain. Bijections set up a perfect pairing between the domain and codomain, and they are the only functions that have a true two-sided inverse. For example, $f(x) = 2x + 3$ from $\mathbb{R}$ to $\mathbb{R}$ is a bijection because every real number is hit exactly once.

What happens to the domain when you divide two functions?

When you form the quotient $(f/g)(x) = f(x)/g(x)$, the domain is the intersection of the domains of f and g , minus any points where $g(x) = 0$. You must exclude those points because division by zero is undefined. For instance, if $f(x) = x + 1$ and $g(x) = x - 3$, both defined on all of $\mathbb{R}$, then $(f/g)(x) = (x+1)/(x-3)$ is defined for all real numbers except $x = 3$.