D'Alembert's Ratio Test of Convergence of Series

In this article, we will formulate the D' Alembert's Ratio Test on convergence of a series.

Let's start.

Statement of D'Alembert Ratio Test

A series $\sum u_n$ of positive terms is convergent if from and after some fixed term $\frac{u_{n+1}}{u_n} < r < 1$, where r is a fixed number. The series is divergent if $\frac{u_{n+1}}{u_n} > 1$ from and after some fixed term.

D'Alembert's Test is also known as the **ratio test of convergence of a series**.

Theorem

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers in R, or a series of complex numbers in C.

Let the sequence a_n satisfy:
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$$

- If $l > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Definitions for Generally Interested Readers

(Definition 1) An infinite series $\sum u_n$ i.e. $\mathbf{u_1 + u_2 + u_3 + \ldots . + u_n}$ is said to be **convergent** if S_n , the sum of its first n terms, tends to a finite limit S as n tends to infinity.

We call S the sum of the series, and write $S = \lim_{n \to \infty} S_n$.

Thus an infinite series $\sum u_n$ converges to a sum S, if for any given positive number ϵ , however small, there exists a positive integer n_0 such that $|S_n - S| < \epsilon$ for all $n \geq n_0$.

(Definition 2)

If $S_n \rightarrow \pm\infty$ as $n \rightarrow \infty$, the series is said to be **divergent**.
Thus, $\sum u_n$ is said to be divergent if for every given positive number λ , however large, there exists a positive integer n_0 such that $|S_n| > \lambda$ for all $n \geq n_0$.

(Definition 3)

If S_n does not tend to a finite limit, or to plus or minus infinity, the series is called **oscillatory**.

Proof & Discussions on Ratio Test

Let a series be $\mathbf{u_1 + u_2 + u_3 + \ldots \ldots \ldots}$ We assume that the above inequalities are true.

From the first part of the statement:

$\frac{u_2}{u_1} < r$, $\frac{u_3}{u_2} < r$ .... where r <1.

Therefore

$$u_1 + u_2 + u_3 + \ldots .$$

$$= u_1(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \ldots.)$$

$$= u_1(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \times \frac{u_2}{u_1} + \ldots.)$$

$$< u_1(1 + r + r^2 + \ldots..)$$

Therefore, $\sum u_n < u_1(1 + r + r^2 + \ldots..)$
or, $\sum u_n < \lim_{n \to \infty} \frac{u_1(1 - r^n)}{1 - r}$

Since r<1, therefore as $n \to \infty,\ r^n \to 0$
therefore $\sum u_n < \frac{u_1}{1 - r}$ =k say, where k is a fixed number.
Therefore $\sum u_n$ is convergent.

Since, $\frac{u_{n+1}}{u_n} > 1$ then, $\frac{u_2}{u_1} > 1$, $\frac{u_3}{u_2} > 1$

Therefore

$u_2 > u_1$

$u_3 > u_2 > u_1$

$u_4 > u_3 > u_2 > u_1$

and so on.

Therefore $\sum u_n = u_1 + u_2 + u_3 + \ldots. + u_n$ > nu_1 .

By taking n sufficiently large, we see that nu_1 can be made greater than any fixed quantity.

Hence the series is divergent.

Academic Proofs

From the statement of the theorem, it is necessary that $\forall n : a_n \neq 0$; otherwise $\dfrac{a_{n+1}}{a_n}$ is not defined.

Here, $\dfrac{a_{n+1}}{a_n}$ denotes either the absolute value of $\dfrac{a_{n+1}}{a_n}$, or the complex modulus of $\dfrac{a_{n+1}}{a_n}$.

Absolute Convergence

Suppose $l < 1$.

Let us take $\epsilon > 0$ such that $l + \epsilon < 1$.

Then:
$\exists N : \forall n > N : \dfrac{a_n}{a_{n-1}} < l + \epsilon$

Thus: (a_n)

$= (\dfrac{a_n}{a_{n-1}} \dfrac{a_{n-1}}{a_{n-2}} \cdots \dfrac{a_{N+2}}{a_{N+1}} a_{N+1})$

$< (l + \epsilon^{n-N-1} a_{N+1})$

By **Sum of Infinite Geometric Progression**, $\sum_{n=1}^{\infty} l + \epsilon^n$ converges.

So by the corollary to the comparison test, it follows that $\sum_{n=1}^{\infty} a_n$ converges absolutely too.

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Divergence

Suppose $l > 1$.
Let us take $\epsilon > 0$ small enough that $l – \epsilon > 1$.
Then, for a sufficiently large N, we have:
$(a_n) =$

$$\left(\frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+2}}{a_{N+1}} a_{N+1}\right)$$

$$> (l – \epsilon^{n-N+1} a_{N+1})$$

But $l – \epsilon^{n-N+1} a_{N+1} \to \infty$ as $n \to \infty$.

So $\sum_{n=1}^{\infty} a_n$ diverges.

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Comments

- When $\frac{u_{n+1}}{u_n} = 1$, the test fails.
- Another form of the test– The series $\sum u_n$ of positive terms is convergent if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}}$ >1 and divergent if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}}$ <1.
- One should use this form of the test in the practical applications.

An Example

Verify whether the infinite series $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \ldots.$ is convergent or divergent.

Solution

We have $u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$ and $u_n = \frac{x^n}{n(n+1)}$

Therefore $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}(1+\frac{2}{n})\frac{1}{x}=\frac{1}{x}$

Hence, when 1/x >1 , i.e., x <1, the series is convergent and when x >1 the series is divergent.

When x=1, $u_n=\frac{1}{n(n+1)}=\frac{1}{n^2}(1+1/n)^{-1}$
or, $u_n=\frac{1}{n^2}(1-\frac{1}{n}+\frac{1}{n^2}-\ldots\ldots)$

Take $\frac{1}{n^2}=v_n$ Now $\lim_{n\to\infty}\frac{u_n}{v_n}=1$, a non-zero finite quantity.

But $\sum v_n=\sum\frac{1}{n^2}$ is convergent.
Hence, $\sum u_n$ is also convergent.