

# Everywhere Continuous Non-differentiable Function

Weierstrass had drawn attention to the fact that there exist functions which are continuous for every value of  $x$  but do not possess a derivative for any value. We now consider the celebrated function given by Weierstrass to show this fact. It will be shown that if

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x) \dots (1)$$
$$= \cos \pi x + b \cos a\pi x + b^2 \cos a^2 \pi x + \dots$$

where  $a$  is an odd positive integer,  $0 < b < 1$  and  $ab > 1 + \frac{3}{2}\pi$ , then the function  $f$  is continuous  $\forall x$  but not finitely derivable for any value of  $x$ .

G.H. Hardy improved this result to allow  $ab \geq 1$ .

We have  $|b^n \cos(a^n \pi x)| \leq b^n$  and  $\sum b^n$  is convergent. Thus, by Weierstrass's  $M$ -Test for uniform Convergence the series [\(1\)](#), is uniformly convergent in every interval. Hence  $f$  is continuous  $\forall x$ .

Again, we have

$$\frac{f(x+h) - f(x)}{h}$$
$$= \sum_{n=0}^{\infty} b^n \frac{\cos[a^n \pi(x+h)] - \cos a^n \pi x}{h} \dots (2)$$

Let, now,  $m$  be any positive integer. Also let  $S_m$  denote the sum of the  $m$  terms and  $R_m$ , the remainder after  $m$  terms, of the series (2), so that

$$\sum_{n=0}^{\infty} b^n \frac{\cos[a^n \pi(x+h)] - \cos a^n \pi x}{h} = S_m + R_m$$

By Lagrange's [mean value theorem](#), we have

$$\frac{|\cos[a^n \pi(x+h)] - \cos a^n \pi x|}{|h|}$$

$$= |a^n \pi h \sin a^n \pi(x + \theta h)| \leq a^n \pi |h|$$

$$|S_m| \leq \sum_{n=0}^{m-1} b^n a^n \pi = \pi \frac{a^m b^m - 1}{ab - 1} < \pi \frac{a^m b^m}{ab - 1}.$$

We shall now consider  $R_m$ .

So far we have taken  $h$  as an arbitrary but we shall now choose it as follows:

We write  $a^m x = \alpha_m + \xi_m$ , where  $\alpha_m$  is the integer nearest to  $a^m x$  and  $-1/2 \leq \xi_m < 1/2$ .

Therefore  $a^m(x+h) = \alpha_m + \xi_m + ha^m$ . We choose,  $h$ , so that

$$\xi_m + ha^m = 1$$

$$\text{i.e., } h = \frac{1 - \xi_m}{a^m} \text{ which } \rightarrow 0 \text{ as } m \rightarrow \infty \text{ for } 0 < h \leq \frac{3}{2a^m} \dots (3)$$

Now,

$$\begin{aligned} a^n \pi(x+h) &= a^{n-m} a^m(x+h) \\ &= a^{n-m} \pi[(\alpha_m + \xi_m) + (1 - \xi_m)] = a^{n-m} \pi(\alpha_m + 1) \end{aligned}$$

Thus

$$\cos[a^n \pi(x+h)] = \cos[a^{n-m}(\alpha_m + 1)\pi] = (-1)^{\alpha_m + 1}.$$

$$\cos(a^n \pi x) = \cos[a^{n-m}(a^m \pi x)]$$

$$= \cos[a^{n-m}(\alpha_m + \xi_m)\pi]$$

$$= \cos a^{n-m} \alpha_m \pi \cos a^{n-m} \xi_m \pi - \sin a^{n-m} \alpha_m \pi \sin a^{n-m} \xi_m \pi$$

$$= (-1)^{\alpha_m} \cos a^{n-m} \xi_m \pi$$

for  $a$  is an odd integer and  $\alpha_m$  is an integer.

Therefore,

$$R_m = \frac{(-1)^{\alpha_m} + 1}{h} \sum_{n=m}^{\infty} b^n [2 + \cos(a^{n-m} \xi_m \pi)] \dots (4)$$

Now each term of series in (4) is greater than or equal to 0 and, in particular, the first

$$\text{term is positive, } |R_m| > \frac{b^m}{|h|} > \frac{2a^m b^m}{3} \dots (3)$$

Thus

$$\left| \frac{f(x+h) - f(x)}{h} \right| = |R_m + S_m| \geq |R_m| - |S_m| > \left( \frac{2}{3} - \frac{\pi}{ab-1} \right) a^m b^m$$

$$\text{As } ab > 1 + \frac{3}{2}\pi$$

therefore  $\left( \frac{3}{2} - \frac{\pi}{ab-1} \right)$  is positive.

Thus we see that when  $m \rightarrow \infty$  so that  $h \rightarrow 0$ , the expression  $\frac{f(x+h) - f(x)}{h}$  takes arbitrary large values. Hence,  $f'(x)$  does not exist or is at least not finite.