

Difference Paradox

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Some of the most fascinating moments in mathematics come not from solving hard problems, but from stumbling into a paradox. A mathematical paradox is a statement (or pair of statements) that seems perfectly logical on the surface, yet leads to a contradiction. You follow every step, nod along, and then realize the conclusion makes no sense. That discomfort is the point. It forces you to look closer and figure out where the reasoning went wrong.

The difference paradox depends on what stays fixed between hypothetical cases. A symbol should not silently change its value inside a single comparison.

The paradox I want to walk you through today is a beautiful example of this. It involves nothing more than two natural numbers and their difference. Both propositions below appear to have valid proofs, yet they directly contradict each other. Your job is to figure out which one is actually correct, and more importantly, why the other one fools you.

Setting Up the Problem

Consider two natural numbers n_1 and n_2 , where one is twice as large as the other. We don't know which one is bigger. It could be $n_1 = 2n_2$, or it could be $n_2 = 2n_1$. All we know is that the doubling relationship exists.

Given this setup, we can state two propositions about the difference between these numbers. Read them carefully, because both sound reasonable at first glance. This is the kind of problem that shows up when studying [set theory](#) and number properties, where precise definitions matter more than intuition.

The Two Propositions

Proposition 1: The difference $n_1 - n_2$, if $n_1 > n_2$, is different from the difference $n_2 - n_1$, if $n_2 > n_1$.

Proposition 2: The difference $n_1 - n_2$, if $n_1 > n_2$, is the same as the difference $n_2 - n_1$, if $n_2 > n_1$.

These two statements are direct opposites. The difference is either the same or it isn't. Both can't be true. Yet both have what look like perfectly valid proofs. Let's examine them.

Proof of Proposition 1

Let $n_1 > n_2$. Since one number is twice the other, this means $n_1 = 2n_2$. Therefore:

$$n_1 - n_2 = 2n_2 - n_2 = n_2$$

So the difference is n_2 .

Now let $n_2 > n_1$. In this case, $n_2 = 2n_1$, which means $n_1 = \frac{1}{2}n_2$. Therefore:

$$n_2 - n_1 = n_2 - \frac{1}{2}n_2 = \frac{1}{2}n_2$$

So the difference is $\frac{1}{2}n_2$.

Since $n_2 \neq \frac{1}{2}n_2$ for any natural number, the two differences are not equal. $\Rightarrow$ Proposition 1 is true. $\square$

The proof of Proposition 1 works because it respects the constraint. When you switch which number is larger, the actual numerical values of n_1 and n_2 change. The difference isn't a fixed property of the pair. It depends on the specific values that satisfy the doubling condition in each case.

Proof of Proposition 2

Let $n_1 > n_2$. Then the difference $n_1 - n_2 = n$, where n is some fixed natural number.

Now if instead $n_2 > n_1$, then $n_2 - n_1$ again equals n .

Therefore, the difference is the same in both cases. $\Rightarrow$ Proposition 2 is true. $\square$

Wait. Both proofs seem valid, but the propositions contradict each other. So which one is actually correct?

Resolving the Paradox

If you're leaning toward Proposition 2, I'd encourage you to slow down and re-read both proofs. Proposition 2 feels intuitively right because we're used to thinking of "the difference between two numbers" as a single fixed quantity. But that intuition is exactly what the paradox exploits.

Let's test it with actual numbers. Take $n_1 = 40$ and $n_2 = 20$. One is twice the other, so our setup holds.

If $n_1 > n_2$: the difference is $n_1 - n_2 = 40 - 20 = 20$.

Now flip the assumption. If $n_2 > n_1$, then $n_2 = 2n_1$, meaning $n_1 = 10$ and $n_2 = 20$. The difference is

$$n_2 - n_1 = 20 - 10 = 10.$$

The differences are 20 and 10. They aren't the same. **Proposition 1 is true.** $\square$

Why This Paradox Works

The trick in Proposition 2's "proof" is subtle and worth understanding. It treats n_1 and n_2 as if their actual values stay the same when you switch which one is larger. But that's not how the problem works.

When we say "one of them is twice the other," the actual values of n_1 and n_2 change depending on which case you're in. In Case 1, where $n_1 > n_2$, you might have $n_1 = 40$ and $n_2 = 20$. In Case 2, where $n_2 > n_1$, the constraint forces different values: $n_1 = 10$ and $n_2 = 20$.

Proposition 2's proof hand-waves over this by saying "the difference is some fixed number n " without acknowledging that the underlying values shift. It assumes the difference is an intrinsic property of the pair, independent of which number is larger. That assumption is false.

This is a common logical trap in mathematics. When you define a quantity using a conditional ("the difference, if $n_1 > n_2$ "), the condition itself constrains the values. Switching the condition doesn't just flip a sign. It changes what the variables actually represent. This connects to the broader discipline of [mathematical problem solving](#), where identifying hidden assumptions is often the hardest step.

Related Paradoxes in Mathematics

The Difference Paradox isn't an isolated curiosity. Mathematics is full of similar traps where intuitive reasoning leads you astray. Here are a few famous examples that exploit related logical gaps.

Grandi's Series. Consider the infinite sum $1 - 1 + 1 - 1 + 1 - 1 + \dots$. If you group the terms as $(1 - 1) + (1 - 1) + \dots$, you get $0 + 0 + \dots = 0$. But if you group them as $1 + (-1 + 1) + (-1 + 1) + \dots$, you get $1 + 0 + 0 + \dots = 1$. The same series appears to equal both 0 and 1. The resolution? The series doesn't converge in the classical sense. The partial sums alternate between 0 and 1 forever, never settling on a value. Guido Grandi published this in 1703, and it sparked decades of debate among mathematicians including Leibniz and Euler about what "sum" really means for infinite series.

Thomson's Lamp. Imagine a lamp that you switch on after 1 minute, off after 1/2 minute, on after 1/4 minute, and so on. After 2 minutes (the geometric series converges), is the lamp on or off? The total time is finite, but you've performed infinitely many switches. Philosopher James F. Thomson proposed this in 1954, and it reveals that not every mathematically described process has a well-defined outcome. The paradox isn't in the math itself but in the assumption that "infinitely many completed actions" must produce a definite state.

The Two Envelope Paradox. You're given two envelopes, one containing twice as much money as the other. You pick one and find \$100. Should you switch? The other envelope has either \$200 or \$50, each with probability $1/2$. The **expected value** of switching seems to be $\frac{1}{2}(200) + \frac{1}{2}(50) = 125$, which is more than \$100. So you should always switch. But the same argument applies regardless of which envelope you pick, which means you should always switch, which is absurd. Like the Difference Paradox, the flaw lies in how the variables get redefined across conditional cases.

All these paradoxes share a root cause: they exploit the gap between informal reasoning and rigorous definitions. When you formalize the concepts properly, using precise definitions of convergence, limits, or conditional probability, the contradictions disappear. The paradox lives in the ambiguity, not in the mathematics.

The Riemann Rearrangement Theorem

The Difference Paradox gets even more interesting when you connect it to one of the most surprising results in real analysis: the Riemann Rearrangement Theorem. This theorem, proved by Bernhard Riemann in 1854, states that if an infinite series is conditionally convergent (converges but not absolutely), you can rearrange its terms to make it converge to any real number you want, or even diverge to infinity.

Take the alternating harmonic series: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$. This converges to $\ln 2 \approx 0.693$. But if you rearrange the terms, taking two positive terms for every one negative term, or three positives for every negative, you can make the same set of numbers add up to any value whatsoever.

The connection to our paradox? Both cases involve the same underlying elements (numbers, terms) producing different results depending on how you arrange or constrain them. The Difference Paradox shows this with two numbers and a doubling constraint. The Riemann Rearrangement Theorem shows it with infinitely many terms and different orderings. The lesson is identical: order and constraints matter as much as the values themselves. Understanding **concepts like supremum and infimum** helps formalize exactly what "convergence" means in these contexts.

Practical Implications

You might think paradoxes like this one are purely academic exercises. They aren't. The logical error behind Proposition 2 shows up in real-world contexts more often than you'd expect.

In numerical computing, floating-point arithmetic can produce different results depending on the order of operations. Adding a million small numbers to a large number gives a different result than adding the small numbers first and then adding the large one. This isn't a paradox in the formal sense, but it's the same underlying issue: the assumption that "the answer" is independent of how you get there.

In statistics, Simpson's Paradox is a direct relative. A trend that appears in several different groups of data can reverse when the groups are combined. A treatment might work better in every subgroup, yet appear worse overall. The "same data" produces opposite conclusions depending on how you condition on variables, exactly like our two propositions.

In physics, the order of measurements matters in quantum mechanics. Measuring position then momentum gives a different result than measuring momentum then position (this is the Heisenberg [Uncertainty Principle](#) in action). The mathematical formalism that captures this, non-commutative operators, exists precisely because physicists learned (the hard way) that the Proposition 2 style of reasoning doesn't work in the quantum world.

In everyday reasoning, people fall for Proposition 2-style logic constantly. "The average salary in this industry is \$80K, so switching companies won't change my earnings." That ignores the fact that your specific position, experience, and location create entirely different conditions in each scenario, just like our n_1 and n_2 take different values under different constraints. This is the same mistake that trips people up when working through [unsolved problems like the Collatz Conjecture](#), where intuitive assumptions about number behavior break down.

How to Spot Flawed Proofs Like This Paradox

The Difference Paradox is a perfect training exercise for developing what mathematicians call "proof hygiene." Here's a checklist you can apply whenever a proof feels suspicious.

Check variable scope. Ask yourself: do the variables mean the same thing at the beginning and end of the proof? In Proposition 2, the variable n is introduced as the difference in one case, then silently assumed to be the same in another case. The symbol stayed the same, but its referent changed. This is the most common source of flawed proofs at every level of mathematics.

Plug in concrete numbers. Abstract proofs can hide errors that concrete examples expose immediately. We used $n_1 = 40$ and $n_2 = 20$ to show the differences were 20 and 10. Always test with at least two different sets of numbers. If you'd tried $n_1 = 100$ and $n_2 = 50$, you'd get differences of 50 and 25. The pattern is clear: the difference in Case 1 is always twice the difference in Case 2.

Identify hidden assumptions. Every proof rests on premises, some stated, some implied. Proposition 2's proof assumes that "the difference" is an intrinsic, case-independent property. That's not stated outright, and it's not true. Training yourself to name unstated assumptions is one of the most valuable skills you can develop in mathematics.

Trace the logic step by step. Don't just check that each step follows from the previous one. Check that each step follows from the *same* premises as the previous one. A proof can have individually valid steps that collectively prove nothing because the premises shift partway through. Proposition 2 is exactly this kind of error.

Ask "what is being held constant?" In any argument involving cases or conditions, something must be fixed for comparison to be meaningful. In our paradox, n_2 was held constant across cases

(it was 20 in both), but n_1 changed from 40 to 10. The proof of Proposition 2 treated both variables as constant, which is where it breaks.

Historical Context: Raymond Smullyan

This paradox comes from Raymond Smullyan, one of the most creative mathematical logicians of the 20th century. Smullyan (1919-2017) was an American mathematician, concert pianist, stage magician, and philosopher who spent his career making logic accessible and entertaining.

He's best known for his puzzle books like *What Is the Name of This Book?* and *The Lady or the Tiger?*, which are packed with self-referential logic puzzles, knights-and-knaves problems, and paradoxes just like this one. His approach was always playful. He wanted you to feel the confusion first, then work your way out of it.

Smullyan taught at institutions including Princeton, Yeshiva University, and Indiana University. His academic work in mathematical logic, particularly on Godel's incompleteness theorems, was serious and rigorous. But he believed that the best way to develop mathematical thinking was through puzzles and paradoxes that challenge your assumptions.

This "curious paradox" about differences is a classic Smullyan construction. It's simple enough that anyone with basic algebra can follow along, yet the logical trap catches even people who should know better. That combination of simplicity and deceptiveness is what made Smullyan's work so effective as a teaching tool.

Key Takeaway

The lesson here goes beyond this specific puzzle. In mathematics, whenever you work with conditional statements, you have to be extremely careful about what changes and what stays fixed when the condition changes.

Proposition 2 fails because it implicitly assumes that the values of n_1 and n_2 remain the same across both cases. They don't. The constraint "one is twice the other" means that switching which number is larger actually changes the numerical values involved.

Here's a good rule of thumb: whenever a proof feels too easy or too clean, check whether the variables mean the same thing in every step. Often, the "proof" quietly redefines what a symbol represents midway through the argument. That's exactly what happens in Proposition 2.

Paradoxes like this one train you to read proofs critically, not just check that each individual step is valid, but verify that the steps are actually connected in the way the author claims. That skill is fundamental to mathematical maturity.

Reference: "A curious paradox", R. Smullyan

The related [OpenStax mathematics textbooks](#) provide additional explanations and exercises for the mathematical definitions used here.

Keeping Hypothetical Cases Separate

The pairs $(n_1, n_2) = (10, 5)$ and $(5, 10)$ both have absolute difference 5. They are relabelings of the same unordered pair. The symbol n_2 has changed value between the 2 cases.

If n_2 is held fixed at 10 instead, the alternatives $n_1 = 20$ and $n_1 = 5$ give positive differences 10 and 5. Holding a named variable fixed defines a different comparison from holding the unordered pair fixed. With an odd natural number, its half would also leave the natural-number domain.

The points to check in this example are:

- State which quantities are fixed before comparing cases.
- Track both the formula and the objects its symbols name.
- A valid algebraic step still needs its domain assumptions.

Check Your Understanding

Try the questions before reading the answers.

Question 1. Do $(10, 5)$ and $(5, 10)$ have the same absolute difference?

Answer. Yes. Both differences are 5.

Question 2. With n_2 fixed at 10, are the differences for $n_1 = 20$ and $n_1 = 5$ equal?

Answer. No. They are 10 and 5.