# Irrational Numbers and
# The Proofs of their Irrationality

A Comprehensive Guide to Classical and Modern Methods

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**Abstract**

How do you prove a number is irrational? You can't just look at a number and declare it irrational. You can't check infinitely many decimal places. You need a proof. And proofs of irrationality are some of the most elegant arguments in all of mathematics.

The basic strategy is always the same: proof by contradiction. Assume the number is rational. Show that assumption leads to something impossible. Conclude the number must be irrational.

This comprehensive guide walks through four different methods, each revealing a different way mathematicians think about this problem. Some date back to ancient Greece. Some are more modern. All of them are beautiful.

## Contents

# 1 What Makes a Number Rational?

Before proving irrationality, we need to be precise about rationality.

**Definition 1.1** (Rational Number). *A rational number is any real number that can be written as a fraction $\frac{a}{b}$ where:*

- *$a$ and $b$ are integers*
- $b \neq 0$
- *$a$ and $b$ are relatively prime (their greatest common divisor is 1)*

That last condition matters. We want the fraction in lowest terms. Without it, you could write $\frac{2}{4}$ and $\frac{1}{2}$ as different representations of the same number. By requiring $\gcd(a, b) = 1$, we get a unique representation.

In formal notation:

$$\frac{a}{b} \in \mathbb{Q} \iff \gcd(a, b) = 1,\ a \in \mathbb{Z},\ b \in \mathbb{Z} \setminus \{0\}$$

## 1.1 What Does "Relatively Prime" Mean?

**Definition 1.2** (Relatively Prime). *Two integers are relatively prime (or coprime) if their greatest common divisor is 1. They share no common factors other than 1.*

**Example 1.3.**

- *2 and 9 are relatively prime*
- *4 and 7 are relatively prime*
- *15 and 49 are relatively prime*
- *But 4 and 6 are* not *relatively prime (they share a factor of 2)*

# 2 What Makes a Number Irrational?

**Definition 2.1** (Irrational Number). *An irrational number is a real number that cannot be expressed as a fraction of two integers, no matter how one tries.*

Famous examples: $\sqrt{2}$, $\sqrt{3}$, $\pi$, $e$, the golden ratio $\phi$.

The decimal expansion of an irrational number goes on forever without repeating. Rational numbers either terminate (like 0.25) or eventually repeat (like $0.333\ldots$). Irrational numbers do neither.

But you can't prove irrationality by computing decimals. You'd need infinitely many. Instead, you need logical arguments.

# 3 The Core Strategy: Proof by Contradiction

Almost every irrationality proof uses the same structure:

1. Assume the number is rational: $x = \frac{a}{b}$ with $\gcd(a, b) = 1$
2. Manipulate this equation algebraically
3. Derive a contradiction (usually that $a$ and $b$ share a common factor)
4. Conclude the assumption was false: $x$ is irrational

The art is in step 2. Different methods give you different tools for that manipulation.

# 4 Method 1: The Pythagorean Approach

This is the oldest and most famous method. The Pythagoreans discovered it around 500 BC, and legend says they were so disturbed by irrational numbers that they tried to suppress the discovery.

## 4.1 Proof That $\sqrt{2}$ Is Irrational

**Theorem 4.1.** *$\sqrt{2}$ is irrational.*

*Proof.* **Step 1:** Assume $\sqrt{2}$ is rational. Then we can write:

$$\sqrt{2} = \frac{a}{b}$$

where $a$ and $b$ are positive integers with $\gcd(a, b) = 1$.

**Step 2:** Square both sides:

$$2 = \frac{a^2}{b^2}$$
$$a^2 = 2b^2$$

**Step 3:** This tells us $a^2$ is even (it equals $2b^2$, which is clearly even).
If $a^2$ is even, then $a$ must be even. (The square of an odd number is always odd.)
So we can write $a = 2k$ for some integer $k$.
**Step 4:** Substitute $a = 2k$ into $a^2 = 2b^2$:

$$(2k)^2 = 2b^2$$
$$4k^2 = 2b^2$$
$$b^2 = 2k^2$$

**Step 5:** Now $b^2$ is also even, which means $b$ is even.
**Step 6:** We've shown both $a$ and $b$ are even. They share a factor of 2. So $\gcd(a, b) \geq 2$.
But we assumed $\gcd(a, b) = 1$. Contradiction!
**Conclusion:** Our assumption was wrong. $\sqrt{2}$ cannot be rational. It is irrational. ∎

## 4.2 An Alternative Argument

*Alternative Proof.* Since $\sqrt{2}$ is positive, assume $a, b > 0$. From the equation $a^2 = 2b^2$, we see that $b$ divides $a^2$.

Now, $b \neq 1$ because that would require $a = \sqrt{2}$, which isn't an integer. So $b > 1$.

By the Fundamental Theorem of Arithmetic, $b$ has at least one prime factor $p$. Since $p \mid b$ and $b \mid a^2$, we have $p \mid a^2$. This means $p \mid a$.

So $p$ divides both $a$ and $b$, meaning $\gcd(a, b) \geq p > 1$.

This contradicts $\gcd(a, b) = 1$. ∎

## 4.3 Generalizing to Other Square Roots

**Theorem 4.2.** *For any positive integer $n$ that is not a perfect square, $\sqrt{n}$ is irrational.*

*Proof.* The same technique applies. Assume $\sqrt{n} = \frac{a}{b}$ with $\gcd(a, b) = 1$.

Then $a^2 = nb^2$.

Any prime $p$ dividing $n$ must divide $a^2$, which means $p \mid a$. Then $p^2 \mid a^2 = nb^2$, so $p \mid b^2$, hence $p \mid b$.

This contradicts $\gcd(a, b) = 1$. ∎

The pattern works because if $p$ is prime and $p \mid a^2$, then $p \mid a$. This is a special property of primes that doesn't hold for composite numbers.

# 5 Method 2: Using the Euclidean Algorithm

This is an elegant variation that uses a powerful theorem about relatively prime numbers.

**Theorem 5.1** (Bézout's Identity)**.** *If* $\gcd(a, b) = 1$*, then there exist integers* $r$ *and* $s$ *such that* $ar + bs = 1$*.*

This is the key insight. Let's use it.

**Theorem 5.2.** $\sqrt{2}$ *is irrational.*

*Proof using Bézout's Identity.* Assume $\sqrt{2} = \frac{a}{b}$ with $\gcd(a, b) = 1$.

By Bézout's Identity, there exist integers $r$ and $s$ with $ar + bs = 1$.

From $\sqrt{2} = \frac{a}{b}$, we have $a = \sqrt{2}b$, so $a^2 = 2b^2$.

Consider the expression $ar + bs = 1$. Multiply both sides by $\sqrt{2}$:

$$\sqrt{2}(ar + bs) = \sqrt{2}$$

Since $a = \sqrt{2}b$, we can write:

$$\sqrt{2} \cdot ar + \sqrt{2} \cdot bs = \sqrt{2}$$

Using $\sqrt{2}a = \frac{a^2}{b} = \frac{2b^2}{b} = 2b$:

$$2br + as = \sqrt{2}$$

The left side is an integer (it's $2br + as$, a sum of integer products).

So $\sqrt{2}$ is an integer? The only integers near $\sqrt{2} \approx 1.414$ are 1 and 2, and $1^2 = 1 \neq 2$ while $2^2 = 4 \neq 2$.

Contradiction. So $\sqrt{2}$ is irrational. ∎

This proof is slick because it doesn't require analyzing even/odd cases. It jumps straight to the contradiction using Bézout's Identity.

# 6 Method 3: Power Series Expansion

Some irrational numbers, like $e$, require more sophisticated techniques. The Pythagorean approach doesn't work directly because $e$ isn't defined as a square root.

Instead, we use the fact that $e$ has a known infinite series expansion:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

## 6.1 Proof That $e$ Is Irrational

This proof is due to Fourier (yes, the same Fourier from Fourier series).

**Theorem 6.1.** *The number* $e$ *is irrational.*

*Proof.* **Assume** $e = \frac{a}{b}$ where $a$ and $b$ are positive integers.

Pick any integer $n > b$ and $n > 1$. Define:

$$N = n! \left( e - \sum_{k=0}^{n} \frac{1}{k!} \right)$$

This is $n!$ times the "tail" of the series for $e$ (everything after the first $n + 1$ terms).

**Step 1: Show $N$ is a positive integer.**

First, $N > 0$ because the tail of the series is positive.

To show $N$ is an integer, write:

$$N = n! \cdot e - n! \sum_{k=0}^{n} \frac{1}{k!}$$

Since $e = \frac{a}{b}$ and $n > b$, we have $n! \cdot e = n! \cdot \frac{a}{b} = \frac{n!}{b} \cdot a$. Since $n > b$, the factor $b$ divides $n!$, so this is an integer.

For the sum, each term $n! \cdot \frac{1}{k!} = \frac{n!}{k!}$ is an integer when $k \leq n$.

So $N$ is the difference of two integers. It's an integer.

**Step 2: Show $N < 1$.**

Expand the tail:

$$N = n! \left( \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \right)$$

$$N = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots$$

Each term is positive, but they decrease rapidly. We can bound this:

$$N < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots$$

This is a geometric series with first term $\frac{1}{n+1}$ and ratio $\frac{1}{n+1}$:

$$N < \frac{1/(n+1)}{1 - 1/(n+1)} = \frac{1}{n} < 1$$

**Step 3: Contradiction.**

$N$ is a positive integer less than 1. But there are no positive integers less than 1!

Contradiction. So $e$ is irrational. ■

This proof is beautiful because it uses the specific structure of $e$'s series expansion. The factorial terms create the perfect setup for the $n!$ multiplier to produce an integer that's too small to exist.

## 7 Method 4: Continued Fractions

Here's a completely different approach. It relies on a powerful theorem about continued fractions.

**Theorem 7.1.** *A real number is rational if and only if its continued fraction representation is finite.*

Equivalently: a number with an infinite continued fraction is irrational.

This gives us an instant irrationality test. Just write the number as a continued fraction. If it goes on forever, the number is irrational.

## 7.1 Examples

**Example 7.2** (The number $e$)**.**

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cdots}}}}}}$$

*The pattern continues: [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...]. It's infinite with a beautiful pattern. So e is irrational.*

*Alternative representations:*

$$e = 1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cdots}}}}$$

$$e = 1 + \cfrac{2}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \cdots}}}}}$$

**Example 7.3** (The number $\pi$)**.**

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}$$

*The continued fraction for $\pi$ is [3; 7, 15, 1, 292, 1, 1, 1, 2, ...]. No discernible pattern, but it's definitely infinite. So $\pi$ is irrational.*

*Alternative representation:*

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cdots}}}}}$$

**Example 7.4** (The number $\sqrt{2}$).

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

*This is [1; 2, 2, 2, 2, . . . ], repeating 2s forever. Infinite, so $\sqrt{2}$ is irrational.*

**Remark 7.5.** *There's something remarkable here. Quadratic irrationals (like $\sqrt{2}$, $\sqrt{3}$, the golden ratio) have eventually periodic continued fractions. Transcendental numbers (like e and $\pi$) have non-periodic infinite continued fractions.*

## 8 Which Method Should You Use?

It depends on the number.

- **For square roots of non-perfect squares:** The Pythagorean approach is fastest and most elementary. You just need basic facts about even/odd numbers and divisibility.
- **For $e$:** The power series method is natural because $e$ is defined by its series. Fourier's proof exploits that definition perfectly.
- **For numbers with known continued fraction expansions:** Just showing the expansion is infinite proves irrationality immediately.
- **The Euclidean algorithm approach:** Elegant but requires knowing Bézout's Identity. It's a good choice if you're already working in number theory.

What about $\pi$? That's harder. The first proof that $\pi$ is irrational came from Johann Lambert in 1761, using continued fractions. Later proofs used integrals and more advanced analysis. There's no simple Pythagorean-style proof for $\pi$.

## 9 A Note on Transcendental Numbers

Proving a number is irrational is one thing. Proving it's transcendental is much harder.

**Definition 9.1** (Transcendental Number). *A transcendental number is a real number that is not algebraic—that is, it is not a root of any non-zero polynomial with rational (or equivalently, integer) coefficients.*

The numbers $\pi$ and $e$ are both transcendental. Proving this took centuries of mathematical development.

- Liouville constructed the first transcendental numbers in 1844
- Hermite proved $e$ is transcendental in 1873
- Lindemann proved $\pi$ is transcendental in 1882 (which also proved that squaring the circle is impossible)

The techniques for proving transcendence are beyond this article. But if you can prove a number is transcendental, you've automatically proved it's irrational. Transcendental implies irrational (but not vice versa).

**Remark 9.2.** $\sqrt{2}$ *is irrational but not transcendental—it's a root of* $x^2 - 2 = 0$.

# 10 Why This Matters

Irrationality proofs teach you something about the nature of mathematical truth. You can't always compute your way to an answer. Sometimes you need pure logic.

The Pythagoreans discovered this 2,500 years ago and it shook their worldview. They believed everything could be expressed as ratios of whole numbers. The existence of $\sqrt{2}$ proved them wrong.

Today, irrational numbers are everywhere:

- Geometry: $\pi$
- Calculus: $e$
- Physics: all over the place
- Music: the equal-tempered scale uses irrational frequency ratios

Understanding how to prove irrationality connects you to one of humanity's oldest mathematical discoveries. And the techniques—proof by contradiction—carry over to countless other areas of mathematics.

# 11 Historical Notes

## 11.1 The Discovery of Irrational Numbers

Legend attributes the discovery of irrational numbers to Hippasus of Metapontum, a Pythagorean, around 500 BC. The Pythagoreans believed that all numbers could be expressed as ratios of whole numbers (integers). The discovery that $\sqrt{2}$ couldn't be expressed this way was a profound shock to their philosophical system.

According to legend, Hippasus was drowned at sea for revealing this discovery, though historians debate whether this actually happened. What's certain is that the discovery fundamentally changed mathematics.

## 11.2 Evolution of Proof Techniques

- **Ancient Greece (500 BC):** Pythagorean proof for $\sqrt{2}$
- **1737:** Euler proves $e$ is irrational
- **1761:** Lambert proves $\pi$ is irrational
- **1844:** Liouville constructs first transcendental numbers
- **1873:** Hermite proves $e$ is transcendental
- **1882:** Lindemann proves $\pi$ is transcendental

# 12 Frequently Asked Questions

### What is an irrational number?

An irrational number is a real number that cannot be expressed as a fraction of two integers. Its decimal expansion goes on forever without repeating. Famous examples include $\sqrt{2}$, $\sqrt{3}$, $\pi$, $e$, and the golden ratio $\phi$. Most real numbers are actually irrational, even though rationals seem more familiar.

### Why can't you prove irrationality by computing decimals?

You'd need infinitely many decimal places. No matter how many digits you compute, you can't rule out the possibility that the pattern eventually starts repeating. A proof must use logic to show that no repeating pattern is possible, not just check a finite number of digits.

### What is proof by contradiction?

You assume the opposite of what you want to prove. Then you show this assumption leads to an impossible situation (a contradiction). Since the assumption causes a contradiction, it must be false. Therefore, the original statement must be true. Almost all irrationality proofs use this technique.

### Who first proved $\sqrt{2}$ is irrational?

The Pythagoreans discovered this around 500 BC. Legend says Hippasus of Metapontum found the proof, and his fellow Pythagoreans were so disturbed they drowned him at sea. The discovery shattered their belief that all numbers could be expressed as ratios of whole numbers.

### What does "relatively prime" mean?

Two integers are relatively prime (or coprime) if their greatest common divisor is 1. They share no common factors besides 1. For example, 8 and 15 are relatively prime because $\gcd(8, 15) = 1$. But 8 and 12 are not because they both share the factor 4.

### How do continued fractions prove irrationality?

There's a theorem: a real number is rational if and only if its continued fraction is finite. So if you can show a number's continued fraction goes on forever, it must be irrational. For example, $\sqrt{2} = [1; 2, 2, 2, \ldots]$ with infinitely many 2s, proving it's irrational.

### Is $\pi$ harder to prove irrational than $\sqrt{2}$?

Much harder. The $\sqrt{2}$ proof uses only basic arithmetic. But $\pi$ isn't defined as a simple algebraic expression, so you can't use the same approach. Johann Lambert first proved $\pi$ irrational in 1761 using continued fractions and advanced analysis. There's no simple Pythagorean-style proof.

### What's the difference between irrational and transcendental?

An irrational number can't be expressed as a fraction. A transcendental number is stronger: it's not a root of any polynomial with integer coefficients. All transcendentals are irrational, but not vice versa. $\sqrt{2}$ is irrational but not transcendental (it's a root of $x^2 - 2 = 0$). $\pi$ and $e$ are transcendental.

### Can you prove all square roots of non-perfect-squares are irrational?

Yes. The same Pythagorean technique works. For $\sqrt{n}$ where $n$ is not a perfect square, assume $\sqrt{n} = \frac{a}{b}$ in lowest terms. Then $a^2 = nb^2$. Any prime $p$ dividing $n$ must divide $a$, and then must divide $b$, contradicting $\gcd(a, b) = 1$. So $\sqrt{n}$ is irrational.

### Who proved $e$ is irrational?

Leonhard Euler proved $e$ is irrational in 1737, though the proof presented in most textbooks is due to Joseph Fourier from the early 1800s. The proof uses $e$'s infinite series expansion and a clever argument showing that a certain quantity would have to be a positive integer less than 1, which is impossible.

## 13 Conclusion

The study of irrational numbers represents one of mathematics' greatest intellectual achievements. From the ancient Pythagoreans' shocking discovery that not all numbers can be expressed as fractions, to modern sophisticated techniques for proving transcendence, the journey reflects mathematics' evolution.

The four methods presented here—the Pythagorean approach, the Euclidean algorithm, power series expansion, and continued fractions—each offer unique insights into the nature of numbers. They demonstrate that mathematics is not merely computation, but a field requiring rigorous logical reasoning and elegant argumentation.

Whether you're a student encountering these proofs for the first time or a mathematician revisiting classical arguments, the beauty of these proofs lies in their simplicity and power. They show us that sometimes, the most profound truths require nothing more than clear thinking and careful logic.

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The existence of irrational numbers reminds us that the universe of mathematics extends far beyond what our intuition suggests. These numbers fill the real line more densely than the rationals, yet proving their existence requires sophisticated arguments that have challenged mathematicians for millennia.